Almost-Cheap Talk without Lying Costs *

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Abstract

This paper considers a sender-receiver game with small signaling costs in Crawford and Sobel (1982) environment, namely *almost-cheap talk* due to Kartik (2005a, 2008). We assume *money burning cost structure*, that is signaling costs are monotonic in message levels, and focus on the situations where the sender and the receiver have common or almost-common interests. We show that in the common interest almost-cheap talk game, there exists no fully separating equilibrium even if signaling costs are sufficiently small. Any monotone equilibrium in this class is a partition equilibrium with finite segments. In the uniform-quadratic setting, there exist a convergent sequence of monotone equilibria whose limit is a fully separating equilibrium in the common interest pure cheap talk game. On the other hand, in the almost-common interest almost-cheap talk game, there exists a fully separating equilibrium, and so does a convergent sequence to a fully separating equilibrium.

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1 Introduction

In this paper, we study a sender-receiver game with small signaling costs in Crawford and Sobel (1982, hereafter CS) environment. Contrary to the standard cheap talk setting, we assume that the sender has to pay positive signaling costs, but they are sufficiently small. Due to the seminal works of Kartik (2005a, 2008), this class of games is called *almost-cheap talk games*. In his works, he assume *lying cost structure*; signaling costs are minimized when the sender tells the truth, but he has to pay more signaling costs when he lies. Moreover, the more his massages are further away from the truth, the more signaling costs he has to bear. Instead of lying cost structure, we assume *money burning cost structure*; that is, signaling costs are monotonic in message levels. Under the environment, we mainly focus on the following two cases; (i) the players have common interests or (ii) they have small conflicts. These situations are called *common interest games* or *almost-common interest games*, respectively. The aims of this paper are to establish how small signaling costs and conflicts between players affect strategic information transmission, and to reexamine the plausibility of fully separating equilibria in common interest pure cheap talk games.

The practical motivation for the study is the difficulty achieving full information transmission even in common interest situations. In the standard CS model, there exists a fully separating equilibrium if the sender and the receiver have common interests, but it is not easy to realize in the real world. We consider, for example, an education game between a professor and a student. The professor has the knowledge of economic theory, and he tries to send it to the student in lectures. After the lectures, the student uses the knowledge in his life. This situation seems to be a common interest case because both the professor and the student would want the student to understand the knowledge better. However, it could be rare that the student completely understands the knowledge. This kind of situation is often observable, but hard to explain by the standard CS model. One potential reason for the divergence is that the standard CS model is quite simplified; for instance, costless communication is an extreme assumption. That is, more or less signaling costs are needed in the reality. In the leading example, economic theory is a complicated knowledge, and then the professor has to bear communication costs (e.g. preparation for the lectures). Therefore, by investigating how these factors affect strategic information transmission, try to figure out what environment can support the difficulty of full communication.

The theoretical motivation comes from the skepticism for focusing on fully separating equilibria in common interest pure cheap talk games. The justification of equilibrium selection seems to be fragile. In the standard CS model, there exists a fully separating equilibrium in common interest cases. However, even in the common interest case, other equilibria still exist; a fully pooling equilibrium, a partition equilibrium, and so on. That is, we face the problem of equilibrium selection. It is well known that the useful criteria such as *intuitive criterion* by Cho and Kreps (1987) or *divinity* by Banks and Sobel (1987) in the costly signaling literature do not work in cheap talk games.[1] Then, several equilibrium selection criteria specialized in cheap talk games are developed.[2] The one of the most suitable criteria for CS environment is *no incentive to separate (NITS) condition*, suggested by Kartik (2005a) and Chen, Kartik, and Sobel (2008, hereafter CKS) . The criterion says that we should select the equilibrium such that the lowest type of the sender weakly prefers the action induced in the equilibrium to the action that is induced when full information is transmitted. One of the justification of NITS is the robustness in the sense of the perturbation by the almost-cheap talk games with lying cost structure. They show that (i) every convergent sequence of monotone equilibria of the almost-cheap talk converges to an equilibrium satisfying NITS, and (ii) for any CS equilibrium satisfying NITS, there exists a convergent sequence of monotone equilibria of the almost-cheap talk whose limit is the CS equilibrium. While they mainly study the situation where the sender and the receiver are conflicting, same results hold also in common interest games. So NITS seems to be a reasonable criterion in common interests cases, and it selects a fully separating equilibrium.

Although both convergence properties (i) and (ii) are doubtless in the situations of conflicting players, there are ad hoc points in common interest cases; especially, (ii) is heavily dependent on the perturbation by lying cost structure. In the common interest almost-cheap talk game, truth-telling is a cost minimizing message for every type of the sender, so the strategy such that every type of the sender reports the true type is a fully separating and cost minimizing strategy. Given the strategy, the receiver can perfectly learn the sender's type by observing messages, and then the receiver always chooses he/her ideal action. Since this is a common interest game, the receiver's ideal action is also the sender' ideal action. Hence, no type of the sender has an incentive to deviate from the strategy; the sender can induce the most preferred action with least signaling costs. This logic holds regardless of the magnitude of signaling costs, so there exists a fully separating equilibrium for any magnitude of signaling costs. Therefore, there exists a convergent sequence, too.

In other words, the almost-cheap talk game with lying costs remains a special structure, namely *richness of cost minimizing messages*, which pure cheap talk games also have, and the structure guarantees the existence of fully separating equilibria in common interest games. Roughly, it is a structure such that every type of the sender has a cost minimizing message, and the set of these

[1] See Banks(1991) and Farrell (1993) for detailed discussion.

[2] See Farrell (1993), Matthews, Okuno-Fujiwara, and Postlewaite (1991), and Rabin (1990).

messages is rich enough so that different types can choose different messages. It is obvious that both pure cheap talk and almost-cheap talk with lying costs have the structure. In pure cheap talk games, any message is a cost minimizing message for any type of the sender, so different types can send different cost minimizing messages. In almost-cheap talk games, the cost minimizing message is truth-telling, so if the type is different, then the cost minimizing message is also different. That is, the perturbation by the almost-cheap talk with lying cost structure does not break the richness of cost minimizing messages of pure cheap talk games, and thus fully separating equilibria still exist. The positive convergence result (ii) seems to be derived by this special structure. Therefore, to reexamine the validity of the results, we introduce the money burning cost structure which does not satisfy the richness of cost minimizing messages; the cost minimizing message is type independent and unique.[3]

Our results are followings. First, there exists no fully separating equilibrium in the common interest almost-cheap talk games even if signaling costs are sufficiently small. Any monotone equilibrium must be a partition equilibrium with finite segments, like the standard CS model. Second, there exists a fully separating equilibrium in the almost-common interest almost-cheap talk games. Hence, we can point out that a small conflict is better than no conflict in terms of information revelation in the class of almost-cheap talk games with money burning cost structure. Finally, in both classes, there exists a convergent sequence of monotonic equilibria whose limit is a fully separating equilibrium in the common interest pure cheap talk game. More precisely, in the almost-common interest almost-cheap talk games, there exist a fully separating equilibrium, and so does a convergent sequence. On the other hand, while the common interest almost-cheap talk games do not have fully separating equilibria, we can find a desired sequence if we focus on the uniform-quadratic model. The results mean whether the perturbed games hold the richness of cost minimizing message does not matter to the convergence. That is, the results can justify focusing on fully separating equilibria in common interest pure cheap talk games.

This paper is structured as follows. The rest of this section discusses related literature. In Section 2, we define the model of almost-cheap talk games with money burning cost structure. In Section 3, we review the results of pure cheap talk games and almost-cheap talk games with lying cost structure, briefly. Common interest almost-cheap talk and almost-common interest almost-cheap talk games are analyzed in Sections 4 and 5, respectively. We will discuss the convergence results in Section 6, and conclude the paper in Section 7.

[3]The cost structure is not novel in the cheap talk literature. See Austen-Smith and Banks (2000) and Kartik (2005b, 2007).

1.1 Related Literature

This work is mostly related to the costly signaling games in the CS environment. One of the standard models is the costly lying model studied by Kartik (2005a, 2008). He incorporates costly signals into the standard CS framework; that is, in addition to usual cheap talk messages, the sender can send also costly reports under the costly lying structure. He characterizes mD1 equilibria in this class, and how the equilibria converge as the signaling costs go to zero. In his setup, there exists a fully separating equilibrium in the common interest cases, but it may or may not exist in conflicting interest cases.[4]

Kartik, Ottaviani, and Squintani (2007, hereafter KOS) study a sender-receiver game in a more general setting. They focus on two behavioral factors, costly lying structure and naive receiver, and show that these factors have same mathematical representations. Furthermore, there exists a fully separating equilibrium even in conflicting interest cases, provided that the state space is unbounded.[5] The unboundedness of the state space is the most different point from Kartik (2005a, 2008).

In contrast to the lying cost structure, Austen-Smith and Banks (2000, hereafter ASB) and Kartik (2005b, 2007) adopt the money burning cost structure; that is, signaling costs are monotonic in message levels. Especially, they assume a linear cost function. Similar to Kartik (2005a, 2008), they study the communication game where the sender can send both costly and costless messages in the CS environment. Their issue is how CS equilibria change if the costly reports are available. They show that the availability of costly reports improves information transmission. Moreover, there exists a fully revealing equilibrium if the costly report space is large enough, but equilibria in this model converge to CS equilibria as the costly report space shrinks. They discuss only common interest cases, but the existence of fully separating equilibria is trivial because of the availability of cheap talk messages.

It is worth while pointing out the differences between this paper and the literature. First, we assume a general money burning cost function. This cost structure is orthogonal to the lying cost structure in Kartik (2005a, 2008) and KOS, and a generalization of liner cost function adopted in ASB and Kartik (2005b, 2007). Second, we mainly focus on common interest and small conflicting interest cases. Most of the literature study conflicting interest cases only, and seldom discuss common interest cases. Furthermore, we also do comparative statics in players conflict, while

[4] The cost structure is crucial for the existence. See Section 3.2.

[5] They do not mention common interest cases, but we can easily check that there exists a fully separating equilibrium in common interest cases because of the same reason mentioned above.

most of the literature fixes the value. Third, in this paper, the sender can send costly signals only; that is, cheap talk messages are not available in our model while, as mentioned above, the literature admits cheap talk messages. The reason why we restrict cheap talk messages is that the availability of cheap talk messages make results trivial, and make the role of costly signals difficult to see because we focus on common interest cases. Finally, we assume bounded state and message spaces contrary to KOS. The unbounded state space is an essential assumption to imply the existence of fully separating equilibria in KOS. However, we show that the money burning cost structure guarantees the existence of fully separating equilibria without rich space assumptions in almost-common interest cases.

As the final comment to the subsection, we remark that our analysis is based on Mailath (1987). He gives us a comprehensive approach to costly signaling models with continuum state space. He characterizes the necessary conditions for the existence of fully separating equilibria.

2 The Model

There exist two players: a sender and a receiver.[6] Let $\Theta \equiv [0,1]$ be the state space and $\theta \in \Theta$ be the realized value of the state of the world, which is known to the sender but unknown to the receiver; that is, it is the sender's private information. We sometimes refer to θ as the sender's type. Let $F(\cdot)$ be a differentiable prior distribution function on the type space Θ with a density function $f(\cdot)$. We assume that $f(\theta) > 0, \forall \theta \in \Theta$. Let $M \equiv [0,1]$ be the message space of the sender and $m \in M$ be a message sent by the sender. Let $A \equiv \mathbb{R}$ be the action space of the receiver and $a \in A$ be an action taken by the receiver. Let $x \geq 0$ be a preference bias, which is a measure of conflict between the sender and the receiver, and smaller x means less conflicting.

We define the sender and the receiver's von Neumann-Morgenstern utility functions, $U^S(a,\theta,x,m), U^R(a,\theta)$ respectively, as follows:

$$U^S(a,\theta,x,m) \equiv u^S(a,\theta,x) - kC(m,\theta), \tag{1}$$

$$U^R(a,\theta) \equiv u^R(a,\theta), \tag{2}$$

where $C(m,\theta)$ represent the signaling cost imposed on the type θ sender who sends the message m, and $k \geq 0$ is a measure of the magnitude of signaling costs. We assume that all information except θ is common knowledge.

We assume the following assumptions on the utility functions:

[6]As a matter of convention, we treat the sender as male and the receiver as female throughout in this paper.

(A1) u^S, u^R, C are all finite, twice continuously differentiable, and each derivative is also finite.

(A2) $u^R(a, \theta) = u^S(a, \theta, 0), \forall a \in A, \forall \theta \in \Theta$.

(A3) $u^S_{11} < 0 < u^S_{12}$ and $u^S_{13} > 0$.[7]

(A4) $u^S_1(a, \theta, x) = 0$ has a unique solution in $a, \forall \theta \in \Theta, \forall x$. Denote it by $a^S(\theta, x)$.

(A5) $a^S_1(\theta, x)$ is finite, $\forall \theta \in \Theta, \forall x$.

(A6) $C(0, \theta) = 0, \forall \theta \in \Theta$.

(A7) $C_1 > 0$ and $C_{11} \geq 0$.

(A1) - (A5) are the assumptions on the standard cheap talk literature.[8] (A1) is a smoothness condition. (A2) means that if $x = 0$, then the sender and the receiver have the same payoff functions. (A3) and (A4) guarantee the single-peaked preferences of the players. Note that by (A2) and (A4), there also exists a unique solution of $u^R_1(a, \theta) = 0$ in $a, \forall \theta \in \Theta$, denoted by $a^R(\theta)$, and $a^R(\theta) = a^S(\theta, 0), \forall \theta \in \Theta$. Also, by (A2) and (A3), $a^R(\theta) < a^S(\theta, x), \forall \theta \in \Theta$ if $x > 0$. (A5) is a technical assumption to avoid the indeterminacy. Note that by (A3), $a^S_1(\theta) > 0, \forall \theta \in \Theta$.

(A6) and (A7) guarantees the money burning cost structure. (A6) means that the least message takes zero cost whatever the sender's type is. In other words, the cost minimizing message is $m = 0$ for any type $\theta \in \Theta$. (A7) assumes that the cost function is monotonic in message levels and it is convex.

The timing of the game is as following. First Nature chooses the sender's type θ according to the distribution function $F(\cdot)$, and only the sender learns it. Second, the sender sends a message m depending on the type θ. Finally, the receiver takes an action a after observing the message m.

The sender's pure strategy is a function $\mu : \Theta \to M$ specifying the message m sent by the type θ. The receiver's pure strategy is a function $\alpha : M \to A$ specifying the action a chosen by the receiver who observes the message m. The receiver's belief, $\mathcal{P} : M \to \Delta(\Theta)$, represents the conditional density function on the type space after observing the message m. The solution concept is Perfect Bayesian Equilibrium (hereafter PBE or just "equilibrium"), and focus on pure strategy equilibria. Moreover, we sometimes restrict our attention to equilibria in which the sender's strategy is nondecreasing in θ. We call them *monotone equilibria*. However, we do not assume it unless explicitly mentioned.

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[7] The subscripts represent partial derivatives.

[8] The uniform-quadratic model used in applied works satisfies the assumptions: $F(\theta) = \theta, u^S(a, \theta, x) = -(a - \theta - b)^2$.

To simplify the following discussion, let us define classes of games that we focus on. A *common interest game* and an *almost-common interest game* are games such that $x = 0$ and small positive x, respectively. Similarly, a *pure cheap talk game* and a *almost-cheap talk game* are games such that $k = 0$, and small but positive k, respectively. Depending on the parameters x, k, there are four possible games. In this paper, we mainly study *common interest almost-cheap talk games* and *almost-common interest almost-cheap talk games.*[9]

Finally, we define the classes of equilibria. For any $\theta \leq \theta'$, let $\bar{a}(\theta, \theta')$ be the receiver's optimal action if she knows that the sender's type lies on $[\theta, \theta']$. That is,

$$\bar{a}(\theta, \theta') \equiv \begin{cases} \arg\max_{a \in A} \int_{\theta}^{\theta'} u^R(a, \tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta} & \text{if } \theta < \theta' \\ a^R(\theta) & \text{if } \theta = \theta' \end{cases} \tag{3}$$

Definition 1 *Let $(\mu^*, \alpha^*; \mathcal{P}^*)$ be a PBE.*

1. *A fully separating equilibrium is an equilibrium in which $\mathcal{P}^*(\theta|\mu^*(\theta)) = 1, \forall \theta \in \Theta$.*

2. *A semi-separating equilibrium is an equilibrium in which there exists a subset $T \subset \Theta$ whose Lebesgue measure is positive such that $\mathcal{P}^*(\theta|\mu^*(\theta)) = 1, \forall \theta \in T$.*

3. *For any $N \in \mathbb{N} \cup \{\infty\}$, a partition equilibrium with N segments is an equilibrium in which there exists a partition of the type space, $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$, such that*

   (a) $U^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, x, m_{i-1}) = U^S(\bar{a}(\theta_i, \theta_{i+1}), \theta_i, x, m_i), \forall i \in \mathbb{N}$ *s.t.* $i < N$,

   (b) $\mu(\theta) = m_i, \forall \theta \in (\theta_i, \theta_{i+1}), \forall i \in \mathbb{N}$ *s.t.* $i < N$ *and* $m_i \neq m_j$ *if* $i \neq j$,

   (c) $\alpha(m_i) = \bar{a}(\theta_i, \theta_{i+1}), \forall i \in \mathbb{N}$ *s.t.* $i < N$.

   Particularly, if $N = \infty$, then the equilibrium is called a partition equilibrium with countably infinite segments.[10]

Intuitively, fully separating equilibria describe the situation where the receiver can perfectly learn the sender's type by observing his messages. In semi-separating equilibria, this property holds partially; that is, the receiver can learn the true type if the sender's type lies on a nontrivial subset T of the state space. We call such a subset T *information revealing set.* In partition equilibria, the receiver can learn which segment the sender belongs to, but still does not know the exact point on the segment. Let us emphasize the difference between semi-separating equilibria and

[9]The other two cases are already discussed in Crawford and Sobel (1982).

[10]A babbling equilibrium is represented by a partition equilibrium with only one segment.

partition equilibria. By definition, an information revealing set has positive Lebesgue measure in semi-separating equilibria, but it has zero measure in partition equilibria; partition equilibria have at most countable segments. In other words, uncountable actions are induced in semi-separating equilibrium, but at most countable actions in partition equilibria. These definitions are a little non-standard because fully and semi-separating equilibria can be thought as partition equilibria with uncountable partitions. However, to avoid confusion, we discriminate separating equilibria between partition equilibria.

3 Benchmarks

For ease of explanation, it is useful to review the main results of the pure cheap talk and the almost-cheap talk with lying costs models, briefly.

3.1 Pure Cheap Talk

Theorem 1 ***(Crawford and Sobel (1982), Theorem 1)*** *Consider the pure cheap talk game with $x > 0$. Then there exists a finite positive integer $N(x)$, and any equilibrium is a partition equilibrium with N segments where $1 \leq N \leq N(x)$. Moreover, for any N with $1 \leq N \leq N(x)$, there exists at least one partition equilibrium with N segments.*

Proof. See CS. ■

If the players have conflicting interests, then fully information transmission is impossible in the standard CS setting. The only information that the receiver can obtain in equilibria is which segments the sender belongs to, and the number of segments is finite. Also, it is well-known that the necessary and sufficient condition for partition equilibria is the existence of a partition, $< \theta_0 \equiv 0, \theta_1, \dots, \theta_N \equiv 1 >$, satisfying condition (a) in Definition 1.

Theorem 2 *Consider the common interest pure cheap talk game. Then, there exists a fully separating equilibrium, but there exist no semi-separating equilibrium and partition equilibrium with countably infinite segments.*

Proof. See the Appendix. ■

On the other hand, if the sender and the receiver have common interests, then a fully separating equilibrium exists; that is, his private information is perfectly transmitted. The logic is straightforward; every type of the sender has no incentive to lie, and the messages are completely credible for the receiver. The interesting part is the non-existence of semi-separating and countably infinite steps partition equilibria. Intuitively, the non-existence comes from the impossibility of finding a type who satisfies the "arbitrage" condition mentioned in (a) in Definition 1. More precisely, because of the common interest and pure cheap talk setting, no type is indifferent to be separating between to be pooling. He always prefers to be separating.

3.2 Almost-Cheap Talk with Lying Costs

Kartik (2005a, 2008) assume $C_{11} > 0 > C_{12}$ and for all θ there exists a unique message $m^S(\theta) \in M$ such that $C_1(m^S(\theta), \theta) = 0$. That is, $m^S(\theta)$ is the sender's cost minimizing message. Because of the convexity, he has to bear more cost if his message is more far away from $m^S(\theta)$. Also note that $m^S(\theta)$ is strictly increasing in θ because of $C_{12} < 0$. In this section, we assume that $m^S(\theta) = \theta, \forall\theta \in \Theta$; that is, truth-telling is a cost minimizing message for any type.[11] Basically, Kartik (2005a, 2008) admit cheap talk messages, but the following results hold even if we prohibit cheap talk messages.

Theorem 3 *(Kartik (2008), Theorem 1) Suppose $x > 0$. Then there exists no fully separating monotone equilibrium in the almost-cheap talk game with lying costs.*

Proof. See Kartik (2008). ■

Intuitively, the impossibility comes from "inflated language". Suppose there exists an equilibrium such that types in the interval $[\theta_l, \theta_h]$ are separating. Then the equilibrium strategy must satisfy $\mu(\theta) > \theta, \forall\theta \in [\theta_l, \theta_h]$. Hence, types sufficiently close to the highest type cannot satisfy the requirement; any available messages have already run out. Therefore there exists no fully separating monotone equilibrium. However, this impossibility is not robust in the sense that it crucially depends on $m^S(1) = 1$ and $M = [0, 1]$. A fully separating equilibria may exist if $m^S(1) < 1$ and the message space is large enough like Austen-Smith and Banks (2000) and Kartik, Ottaviani and Squintani (2007).

Theorem 4 *(Kartik (2008) p.15) There exists a fully separating equilibrium in the common interest almost-cheap talk games with lying costs.*

Proof. See Kartik (2008). ■

The construction is straightforward as mentioned in Section 1; a strategy $\mu(\theta) = \theta, \forall\theta \in \Theta$ is fully separating and cost minimizing. Given $\mu(\cdot)$, the receiver's best response is $\alpha(\mu(\theta)) = a^R(\theta)$.

[11] Kartik (2005a, 2008) assumes more general framework to discuss wider class of communication, but the setting is enough in this paper.

Since $x = 0$, $a^S(\theta, 0) = a^R(\theta), \forall \theta \in \Theta$. Therefore, any type has no incentive to deviate because every type of the sender can induce his ideal action with minimized cost.

3.3 Richness of Cost Minimizing Messages

The reminder of the section, we discuss the special structure, richness of cost minimizing messages, that is shared by both the pure cheap talk and the almost-cheap talk with lying costs models.

Definition 2 *A sender-receiver game satisfies richness of cost minimizing messages if (i) for every type of the sender there exists a cost minimizing message, $m^S(\theta)$, and (ii) let $M^S \equiv \{m^S(\theta)|\theta \in \Theta\}$ be the set of cost minimizing messages. Then there exists a one-to-one function $\phi : \Theta \to M^S$ subject to $\phi(\theta) = m^S(\theta), \forall \theta \in \Theta$.*

Lemma 1 *If a common interest sender-receiver game satisfies the richness of cost minimizing messages, then there exists a fully separating equilibrium.*

Proof. See the Appendix. ■

The richness of cost minimizing messages is the sufficient condition for the existence of fully separating equilibrium in common interest sender-receiver games. It is obvious that both the pure cheap talk and the almost-cheap talk with lying costs satisfy the property. In the pure cheap talk games, for every type θ, any message m is a cost minimizing message. Hence, $M^S = M$, and there are many one-to-one functions. In the almost-cheap talk games with lying costs, truth-telling is a unique cost minimizing message. Then $M^S = M$ and there exists a desired function ϕ defined by $\phi(\theta) = \theta, \forall \theta \in \Theta$. However, the almost-cheap talk with money burning costs does not satisfy the richness of cost minimizing messages; that is, $m^S(\theta) = 0, \forall \theta \in \Theta$ and then $M^S = \{0\}$. Thus, we cannot construct the desired one-to-one function. In other words, the existence of fully separating equilibria in the games studied in this paper is not trivial. In the rest of the paper, we refer to the almost-cheap talk games with money burning costs as, simply, "almost-cheap talk games".

4 Common Interest Almost-Cheap Talk Game

4.1 No Fully Separating Equilibrium

Now, we assume that $k > 0$ but sufficiently small. As long as $k > 0$, we can use the results in the costly signaling literature. The following theorem is the necessary condition for the existence of fully separating equilibria. This is based on Mailath (1987, Theorem 1).

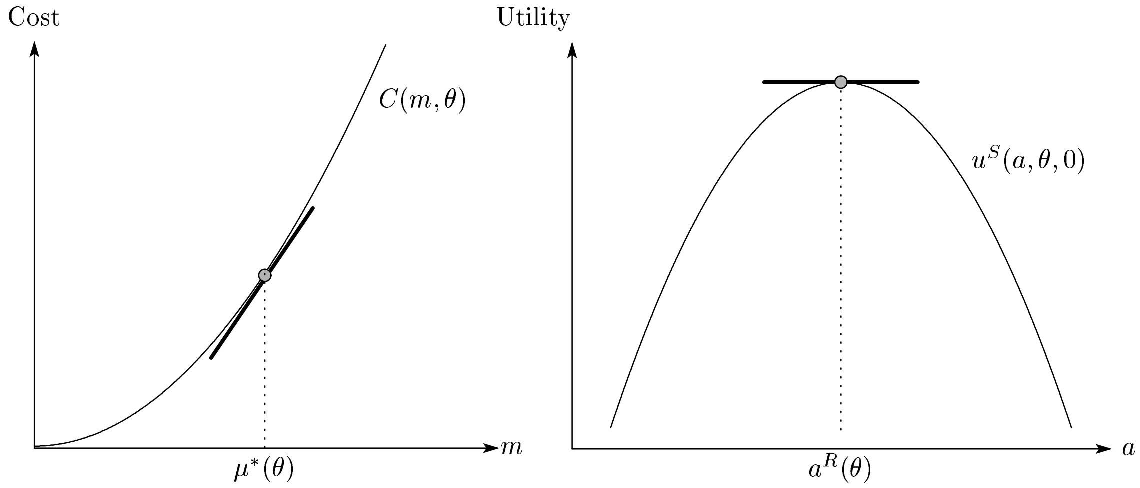

Figure 1: The sender's downward biased preference

Theorem 5 *Suppose $\mu(\cdot)$ is a fully separating equilibrium strategy. Then $\mu(\cdot)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. Moreover, where it is differentiable, it satisfies;*

$$\mu_1(\theta) = \frac{u_1^S(a^R(\theta), \theta, x)a_1^R(\theta)}{kC_1(\mu(\theta), \theta)}. \tag{4}$$

Proof. See the Appendix. ■

Note that the fully separating equilibrium strategy $\mu(\cdot)$ is one-to-one on $[0,1]$. Let $\mu([0,1])$ be the set of equilibrium messages associated with the fully separating strategy $\mu(\cdot)$. Then, once the differentiability of μ is guaranteed, it is characterized by

$$\mu(\theta) \in \arg\max_{m \in \mu([0,1])} u^S(a^R(\mu^{-1}(m)), \theta, x) - kC(m, \theta).[12] \tag{5}$$

By the first order condition with respect to m and evaluating at $m = \mu(\theta)$, we can obtain the ordinal differential equation (4). Now, we assume $x = 0$ until the end of this section. By applying this theorem, we obtain the following impossibility results;

Proposition 1 *There exists no fully separating equilibrium in the common interest almost-cheap talk game.*

Proof. See the Appendix. ■

Intuitively, this impossibility comes from the sender's downward biased preference endogenously generated by the signaling cost. See Figure 1. Suppose $\mu^*(\cdot)$ is a fully separating equilibrium strategy. Since $\mu^*(\cdot)$ should be one-to-one, almost all types pay positive signaling costs. Moreover,

[12]Since $\mu(\cdot)$ is continuous, $\mu([0,1])$ should be an interval.

it has to be continuous, and then for any sufficiently small $\epsilon > 0$, there exists a type θ' on the neighborhood of type θ such that $\mu^*(\theta') = \mu^*(\theta) - \epsilon$. Suppose $\mu^*(\theta) > 0$ and consider the deviation of type θ to $\mu^*(\theta) - \epsilon$ for sufficiently small $\epsilon > 0$. By (A7), this deviation strictly reduces the signaling cost that he has to pay. On the other hand, the slope of u^S at $(a^R(\theta), \theta, 0)$ is 0. Then this deviation induces an action that gives type θ almost same utility that he obtains when he sends $\mu^*(\theta)$. Therefore, the cost reducing effect dominates the loss of inducing undesired action; that is, almost all types have an incentives to mimic sufficiently smaller types than themselves. It is an endogenous downward bias. Note that the almost-cheap talk games with lying cost structure do not generate the downward bias.

We can also show that there exists no semi-separating equilibrium in this class of games due to the same logic. By the definition, there exists a positive measured information revealing set T on semi-separating equilibria. Instead of entire type space, we restrict our attention to the information revealing set T, and apply the same logic.

Corollary 1 *There exists no semi-separating equilibrium in the common interest almost-cheap talk game.*

Proof. See the Appendix. ■

4.2 Characterization of Equilibria

In the previous subsection, we show that there exists no fully and semi-separating equilibrium in the common interest almost-cheap talk games. Then on the equilibrium path, at most countable actions are induced. The reminder of the section, we focus on monotone equilibria, and also put an additional assumption on the cost function;

(B1) $C_{12} \leq 0$.

It is a kind of single-crossing condition; that is, the marginal cost of message is non-decreasing in type.[13] However, we do not assume both the monotonicity and (B1) condition unless we explicitly mentioned.

In order to characterize equilibria in this class, we show some properties;

Lemma 2 *In any equilibrium in the common interest almost-cheap talk game, the two different types θ and θ' induce the same action if and only if $\mu(\theta) = \mu(\theta')$.*

[13]This assumption is commonly used in the literature of the mixed model of cheap talk and costly signaling. See Austen-Smith and Banks (2000), Kartik (2005a, 2005b, 2007, 2008), and Eső and Schummer (2007).

Proof. See the Appendix. ■

Contrary to the usual cheap talk literature, the message which induces an action a is unique. That comes from the positive signaling cost. For example, in a partition equilibrium of pure cheap talk games, types on the same segment do not necessary send the same message; they could randomize on some message set or use a different message to induce the same action because any message is completely costless.

Lemma 3 *In any monotone equilibrium in the common interest almost-cheap talk game, the set of types that induces the same action is an interval.*

Proof. See the Appendix. ■

This result is based on Kartik (2005a, Lemma 1). If we restrict our attention to monotone equilibria, then the entire type space is partitioned by intervals. This is the reason why we focus on monotone equilibria. Let $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ be a partition of a monotonic equilibrium, and call each θ_i except for θ_0, θ_N a boundary type of the partition.

Lemma 4 *In any monotone equilibrium except the babbling equilibrium, any boundary type θ_i of the partition $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ should satisfy the following boundary conditions; for all $i \in \mathbb{N}$ such that $i < N$,*

$$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) - kC(m_{i-1}, \theta_i) = u^S(\bar{a}(m_i, \theta_{i+1}), \theta_i, 0) - kC(m_i, \theta_i) \quad (6)$$

where $m_{i-1} = \mu(\theta), \forall \theta \in (\theta_{i-1}, \theta_i)$ and $m_i = \mu(\theta'), \forall \theta' \in (\theta_i, \theta_{i+1})$.

Proof. See the Appendix. ■

Intuitively, this lemma means that for the boundary types, it is indifferent between to be pooling with their left interval and the right interval. That is, (6) is just the *arbitrage condition* in Crawford and Sobel (1982). Therefore, combining this lemma with the previous one, we can conclude that any monotone equilibrium in the common interest almost-cheap talk games is a partition equilibrium.

Also, there exists no partition equilibrium with countably infinite segments with monotone messages (hereafter, monotone partition equilibrium with countably infinite segments) in the common interest almost-cheap talk games. The non-existence result also holds in the common interest pure cheap talk games, but the reason is different. In the common interest pure cheap talk games, any type cannot be the boundary type between a singleton and a positive measured segments because there is no action satisfying the arbitrage condition due to the costless communication. On the other hand, in the almost-cheap talk games, the sender has to pay signaling costs, and then the action-message pair being indifferent for the boundary type could exist by choosing message appropriately. However, it is impossible because of the sender's downward bias that we mentioned in the previous subsection. This endogenous bias has a similar effect to the exogenous bias $x > 0$ in the standard cheap talk literature.

Proposition 2 *In the common interest almost-cheap talk game, there exists no monotone partition equilibrium with countably infinite segments.*

Proof. See the Appendix. ■

Conversely, if we assume the single-crossing condition (B1), then we obtain a sufficient condition for the existence of partition equilibria in this class. These results are summarized in the following proposition. They are similar to the results of Crawford and Sobel (1982).

Proposition 3 *Consider the common interest almost-cheap talk game.*

1. *Any monotone equilibrium is a partition equilibrium with N segments where $N \in \mathbb{N}$. That is, in any monotone equilibrium, there exists an equilibrium partition $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ and equilibrium messages $\{m_0, \ldots, m_{N-1}\}$ such that*

   (a) *For all $i = 1, 2, \ldots, N-1$,*

   $$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) - kC(m_{i-1}, \theta_i) = u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta_i, 0) - kC(m_i, \theta_i) \tag{7}$$

   (b) *$\mu(\theta) = m_i, \forall \theta \in (\theta_i, \theta_{i+1})$ for all $i = 0, \ldots N-1$, and $m_i \neq m_j$ for $i \neq j$,*

   (c) *$\alpha(m_i) = \bar{a}(\theta_i, \theta_{i+1})$ for all $i = 0, \ldots, N-1$.*

2. *Suppose (B1). Then, conversely, if there exist a partition $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ and a profile of messages $\{m_0 \equiv 0, \ldots, m_{N-1}\}$ with $m_i < m_{i+1}$ for $i = 0, \ldots, N-2$ satisfying (a), then there exists a partition equilibrium with N segments.*

Proof. See the Appendix. ■

Next, we explain the existence of the equilibria. From the previous result, the sufficient condition for the existence of the partition equilibrium with N segments is the existence of a partition $< \theta_0 \equiv 0, \ldots, \theta_N \equiv 1 >$ and a monotone message profile $\{m_0 \equiv 0, \ldots, m_{N-1}\}$. Under additional assumptions, the existence of them is guaranteed by the results of Crawford and Sobel (1982) when the magnitude of signaling cost k is sufficiently small. For $i = 1, \ldots, N-1$, let

$$W_i(\theta_1, \theta_2, \ldots, \theta_{N-1}; k) \equiv u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) - u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta_i, 0) + k\Big(C(\theta_i, \theta_i) - C(\theta_{i-1}, \theta_i)\Big). \tag{8}$$

If there exists a partition equilibrium with N segments in the common interest pure cheap talk game, then the equilibrium partition $< \theta_0^* \equiv 0, \theta_1^*, \ldots, \theta_N^* \equiv 1 >$ should satisfy the arbitrage condition; for $i = 1, \ldots, N-1$,

$$u^S(\bar{a}(\theta_{i-1}^*, \theta_i^*), \theta_i^*, 0) = u^S(\bar{a}(\theta_i^*, \theta_{i+1}^*), \theta_i^*, 0). \tag{9}$$

That is, for $i = 1, \ldots, N-1$, $W_i(\theta_1^*, \theta_2^*, \ldots, \theta_{N-1}^*; 0) = 0$. Then, let

$$J \equiv \begin{pmatrix} \partial W_1/\partial\theta_1 & \partial W_1/\partial\theta_2 & \ldots & \partial W_1/\partial\theta_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial W_{N-1}/\partial\theta_1 & \partial W_{N-1}/\partial\theta_2 & \ldots & \partial W_{N-1}/\partial\theta_{N-1} \end{pmatrix}. \tag{10}$$

Proposition 4 *Suppose that there exists a partition equilibrium with N segments whose equilibrium partition is* $< \theta_0^* \equiv 0, \theta_1^*, \ldots, \theta_N^* \equiv 1 >$ *in the common interest pure cheap talk game, and the determinant of the Jacobian J at* $(\theta_1^*, \ldots, \theta_{N-1}^*; 0)$ *is not zero. Then there exists a partition with N segments and a monotone message profile* $\{m_0 \equiv 0, m_1, \ldots, m_{N-1}\}$ *satisfying the boundary conditions (7) in the common interest almost-cheap talk game if k is sufficiently close to* 0.

Proof. See the Appendix. ■

We have some remarks. First, this is an application of the Implicit Function Theorem. Under these assumptions, we can find a partition of the type space and a profile of monotone messages with $m_i = \theta_i$ for $i = 0, \ldots, N-1$ satisfying (7) if the magnitude of signaling cost $k > 0$ is sufficiently close to 0. Second, the requirements of Proposition 4 seems to be demanding, but the conditions hold in the uniform-quadratic model. Hence, we obtain the following corollary.

Corollary 2 *Suppose the uniform-quadratic model and (B1). Then for any natural number N, there exists a partition equilibrium with N segments in the common interest almost-cheap talk game.*

Proof. See the Appendix. ■

5 Almost-Common Interest Almost-Cheap Talk Games

In this section, we assume both $k > 0$ and $x > 0$ but they are small. First, we check whether there exists fully separating equilibria in this class. This is also a costly signaling model, and then the necessary condition for the existence of fully separating equilibria is given by the differential equation (4). Moreover, if we put an initial condition, $\mu(0) = 0$, then we obtain the more detailed characterization. This result is almost same to Kartik (2005a, Lemma 5) with trivial change.

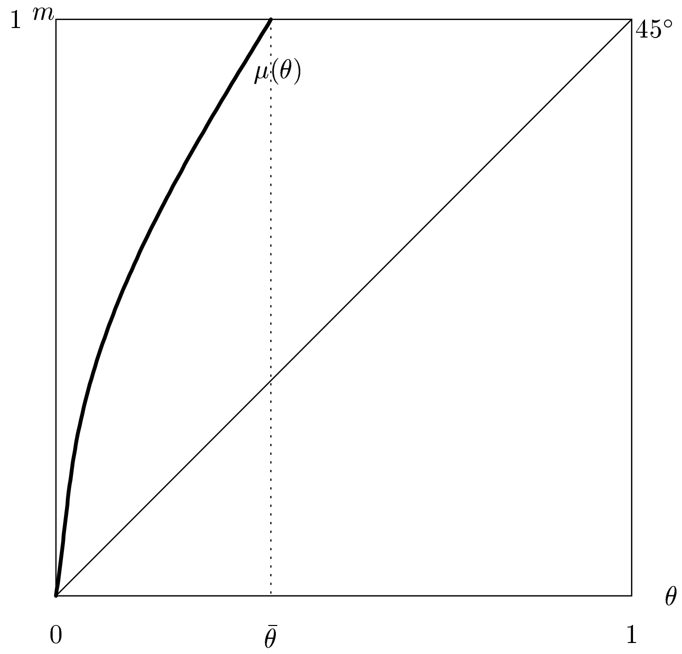

Figure 2: An example of $\mu(\theta)$

Lemma 5 *In the almost-common interest almost-cheap talk game, there exists a unique* $\bar{\theta} \in [0, 1]$ *and* $\mu : [0, \bar{\theta}] \to [0, 1]$ *s.t.*

1. $\mu(\cdot)$ *is strictly increasing and continuous on* $[0, \bar{\theta}]$,

2. $\mu(0) = 0$,

3. μ *solves (4) on* $(0, \bar{\theta})$, *and*

4. $\mu(\bar{\theta}) = 1$ *if* $\bar{\theta} < 1$.

Proof. See the Appendix. ■

Figure 2 represents an example of $\mu(\cdot)$ mentioned in Lemma 5. This lemma means that under the appropriate parameters, there could exist a semi-separating equilibrium with the information revealing set $[0, \tilde{\theta}]$ for $\tilde{\theta} \leq \bar{\theta}$.[14] Moreover, given the preference bias $x > 0$, $\bar{\theta}$ goes to 0 as k goes to 0 because μ_1 close to infinity and the message space M is bounded. That is, the semi-separating equilibrium hardly exists if k is sufficiently close to 0 under the fixed $x > 0$.

However, if x is sufficiently small, then the result is drastically changed even if k is small. As we can see from (4), the solution to this problem depends on the parameters x, so hereafter we

[14]The boundary type of the information revealing set and the pooling set has to be indifferent between revealing his type and pooling. Therefore, in general, $\tilde{\theta}$ is less than $\bar{\theta}$.

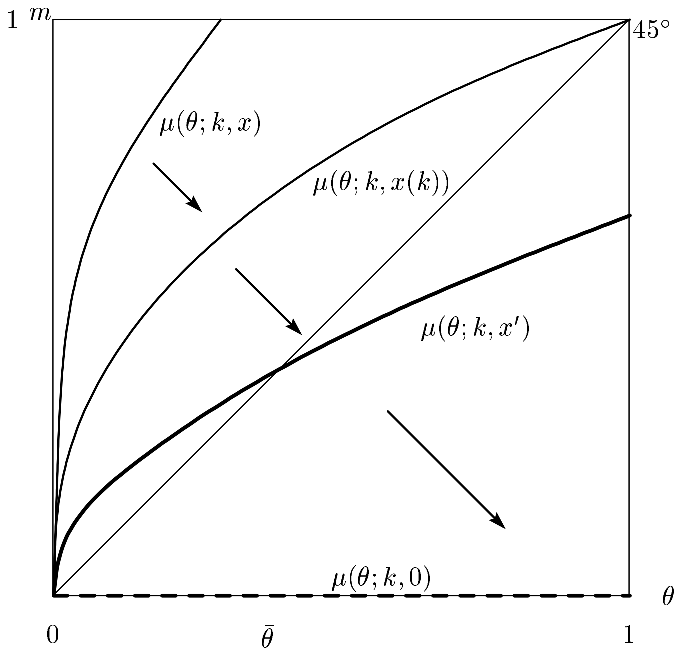

Figure 3: Proposition 5

represent the solution to this problem by $\mu(\theta; x)$ and $\bar{\theta}$ is also represented by $\bar{\theta}(x)$. That is, the initial value problem becomes

$$\mu_1(\theta; x) = \frac{u_1^S(a^R(\theta), \theta, x)a_1^R(\theta)}{kC_1(\mu(\theta; x), \theta)}, \mu(0; x) = 0. \tag{11}$$

Then, we obtain the following result.

Proposition 5 *For any $k > 0$, there exists $x(k) > 0$ such that $\forall x \in (0, x(k))$, $\bar{\theta}(x) = 1$. That is, there exists a fully separating equilibrium under the sufficiently small $x > 0$.*

Proof. See the Appendix. ■

Roughly, since the solution of the initial value problem (11) is pointwise increasing in x, we can find some cutoff value $x(k)$ such that for all $x \leq x(k)$, $\bar{\theta}(x) = 1$ as shown in Figure 3. That is, even if we do not assume a large message space, like Austen-Smith and Banks (2000) and Kartik, Ottaviani, and Squintani (2007), fully separating equilibria exist when x and k are sufficiently small.

We can conclude that in the almost-cheap talk games with money burning costs, a small conflict between players can be better than no conflict in terms of information revelation. This implication is completely opposite to the standard CS and the almost-cheap talk with lying costs models; in these models, a small conflict always improves information transmission. Intuitively, a small conflict makes the sender's deviations costly; that is, mimicking a sufficiently smaller type reduces the signaling costs, but the loss from inducing the action for the smaller type could dominate the gain from the cost reducing effect. Therefore, we can construct fully separating equilibria by choosing messages appropriately.

6 Convergence Results

Given the results, in this section, we consider how sequences of equilibria behave when the magnitude of signaling costs k and the preference bias x goes to 0. Especially, we focus on whether there exists a convergent sequence to a fully separating equilibrium in the common interest pure cheap talk games. Define a function $\beta : \Theta \to A$ by $\beta = \alpha \circ \mu$, and call it an *equilibrium outcome.*

6.1 Almost-Common Interest Almost-Cheap Talk

First, we study the almost-common interest almost-cheap talk games. By Proposition 5, there exists a fully separating equilibrium for any $k > 0$ if x is sufficiently small, so does a convergent sequence. Let β^{kx} and β^* be the equilibrium outcome in almost-common interest almost-cheap talk game and the fully separating equilibrium outcome in the common interest pure cheap talk game, respectively.

Proposition 6 *There exists a convergent sequence of equilibria in the almost-common interest almost-cheap talk game such that $\beta^{kx} \to \beta^*$ pointwise as $k \to 0$ and $x \to 0$.*

Proof. See the Appendix. ■

Consider, for example, the uniform-quadratic model: $u^S(a, \theta, x) - kC(m, \theta) = -(a - (\theta + x))^2 - km$. The solution for the initial value problem (11) in this special case is $\mu(\theta; x) = \dfrac{2x}{k}\theta$, and then $x(k) = \dfrac{k}{2}$. The shaded region in Figure 4 represents the set of pairs of x, k that can support a fully separating equilibrium. Therefore, by choosing x, k in this shaded region, we can construct a convergent sequence of equilibrium whose limit is a fully separating equilibrium as both $k \to 0$ and $x \to 0$.

6.2 Common Interest Almost-Cheap Talk: Uniform-Quadratic Model

Next, consider the common interest almost-cheap talk games. By Proposition 1, there exists no fully separating equilibrium. However, as long as we restrict our attention to the uniform-quadratic

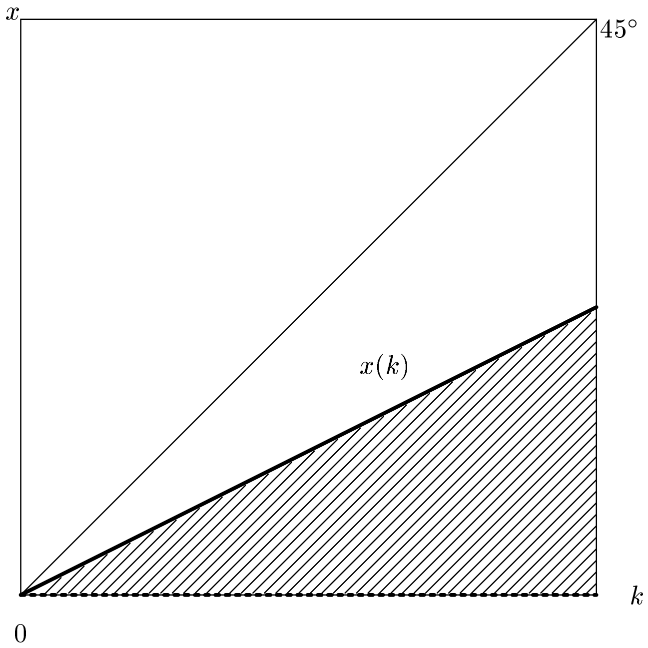

Figure 4: An example pf Proposition 6

model, we can conclude that there exists a convergent sequence as $k \to 0$.

Proposition 7 *Suppose the uniform-quadratic model and (B1). Then, there exists a sequence of monotone equilibria in the common interest almost-cheap talk game whose limit is a fully separating equilibrium in the common interest pure cheap talk game as* $k \to 0$.

Proof. See the Appendix. ■

Since for any natural number N there exists a partition equilibrium with N segments when k is sufficiently small, so does the convergent sequence to a fully separating equilibrium. We have some remarks; first, in the neighborhood of $k = 0$, the magnitude of signaling costs k has similar effects to the preference bias x. In other words, as long as these numbers are positive, an equilibrium must be a partition equilibrium with finite segments. However, as these parameters go to zero, finer partition equilibria exist, and finally the sequence converges to a fully separating equilibrium. However, the biased direction is opposite; x generates upward biases, but k generates downward biases as mentioned in Section 4.

Second, while the almost-cheap talk games with money burning costs do not satisfy the richness of cost minimizing messages, there exists a convergent sequence to a fully separating equilibrium. That is, the convergence result does not depend on whether the perturbed games satisfy the richness of cost minimizing messages. Hence, fully separating equilibria is robust in this sense, so focusing on these equilibria can be justified more by the result.

7 Conclusion

In this paper, we consider the almost-cheap talk games with money burning costs, and characterize equilibria in both the common interest and almost-common interest cases. In almost-common interest games, there exists a fully separating equilibrium for any positive magnitude of costly signaling, and so does the convergent sequence to a fully separating equilibrium. On the other hand, there exists no fully separating equilibrium in the common interest cases because of the downward bias generated by money burning cost structure. The downward bias has a similar effect to the exogenous preference bias in the standard CS model. Therefore, we obtain the counter intuitive result that a small conflict is more useful than no conflict in terms of information revelation. Moreover, in the quadratic-uniform model, there exists a convergent sequence to a fully separating equilibrium as an analogous to the convergence of preference bias in the CS model. That is, whether the perturbed games satisfies the richness of cost minimizing messages does not matter to the convergence result, contrary to the observation mentioned in Section 1. Thus, in this sense, focusing on fully separating equilibria in common interest games is justified more by the results, at least in the uniform-quadratic model. We believe that same result should hold in general settings, but it is left for the future works.

Appendix

Proof of Lemma 1

Suppose a sender-receiver game satisfies the richness of cost minimizing messages. Then there exists a one-to-one function $\phi : \Theta \to M^S$. Define the sender's strategy $\mu(\cdot)$ by $\mu(\theta) = \phi(\theta), \forall\theta \in \Theta$. Since it is one-to-one, the receiver's optimal action is $\alpha(\mu(\theta)) = a^R(\theta)$. Since it is a common interest game, $a^R(\theta) = a^S(\theta, 0), \forall\theta \in \Theta$. That is, whatever off equilibrium path belief the receiver has, the sender has no incentive to deviate because he can induce the most desired action with least cost. ■

Proof of Theorem 2.

1. Show that the following is a PBE;

$$\begin{aligned} \mu^*(\theta) &= \theta, \forall\theta \in \Theta, \\ \alpha^*(m) &= a^R(m), \forall m \in M, \\ \mathcal{P}^*(\theta|m) &= \begin{cases} 1 & \text{if } \theta = m \\ 0 & \text{Otherwise} \end{cases} \end{aligned} \tag{12}$$

Given $\mathcal{P}^*$, α^* is a best response of the receiver. Then check the optimality of μ^* given α^*. Note that since $x = 0$, $a^R(\theta) = a^S(\theta, 0), \forall\theta \in \Theta$. That is, by following μ^*, the sender can induce his ideal action. Then any type has no incentive to deviate. Therefore, $\mathcal{P}^*$ is consistent with Bayes' rule. Thus, it is a PBE, and a fully separating equilibrium. ■

2. Suppose, by way of contradiction, that there exists a semi-separating equilibrium. Let θ be the boundary point of the information revealing set. Let a' be the equilibrium action induced by types in $(\theta, \theta + \delta)$ for some $\delta > 0$. [15]

Claim 1 *In the semi-separating equilibrium, the following condition should hold;*

$$u^S(a^R(\theta), \theta, 0) = u^S(a', \theta, 0). \tag{13}$$

Proof. Suppose, by way of contradiction, that (13) does not hold on the equilibrium. Then $u^S(a^R(\theta), \theta, 0) > u^S(a', \theta, 0)$ should hold because $a^R(\theta)$ is the ideal action for type θ. Then there exists $\epsilon > 0$ such that the types who belong to $(\theta, \theta + \epsilon)$ have incentives to mimic type θ, a contradiction. □

However, since $a^R(\theta)$ is a unique ideal action for type θ, (13) never holds, a contradiction. ■

3. Suppose, by way of contradiction, that there exists a partition equilibrium with countably infinite segments. Since the type space is bounded, at least one trivial segment, $\{\theta\}$, exists, and its adjacent segments should have positive measure; otherwise, these two segments construct an information revealing set whose Lebesgue measure is positive. Without loss of generality, denote it by $[\theta', \theta]$. The following boundary condition should hold;

$$u^S(a^R(\theta), \theta, 0) = u^S(\bar{a}(\theta', \theta), \theta, 0). \tag{14}$$

[15] We can obtain the same result if we define the set of types who induces the action a' as $(\theta - \delta, \theta)$.

However, because $x = 0$, $a^R(\theta) = a^S(\theta, 0), \forall \theta \in \Theta$, and then $a^R(\theta)$ is a unique ideal action of the type θ sender. Thus, (14) never hold, a contradiction. ■

Proof of Theorem 5

To prove this theorem, we need some lemmas.

Lemma 6 *Suppose $\mu(\cdot)$ is an equilibrium strategy and one-to-one on the positive measured subset $T \subseteq \Theta$. Then $\mu(\cdot)$ should be continuous on T.*

Proof. The proof is due to Kartik (2008, Claim in Lemma 1) with trivial change. Suppose, by way of contradiction, that μ has a discontinuity point at $\theta' \in T$. Take a convergent sequence $\theta_n \in T$ whose limit is θ'. Since the message space M is compact, there exists a convergent subsequence $\mu(\theta_n) \in M$ with its limit $\bar{\mu}$. There are two cases; (i) $\bar{\mu} > \mu(\theta')$, or (ii) $\bar{\mu} < \mu(\theta')$.

(i) By (A7), $C(\bar{\mu}, \theta_n) > C(\mu(\theta'), \theta_n)$. Then for sufficiently large n, $C(\mu(\theta_n), \theta_n) > C(\mu(\theta'), \theta_n)$ due to the continuity of C. On the other hand, $u^S(a^R(\theta_n), \theta_n, x) - u^S(a^R(\theta'), \theta_n, x)$ is arbitrarily close to 0 if n is sufficiently large because u^S, u^R, a^R are all continuous. That is, for sufficiently large n,

$$u^S(a^R(\theta_n), \theta_n, x) - kC(\bar{\mu}, \theta_n) < u^S(a^R(\theta'), \theta_n, x) - kC(\mu(\theta'), \theta_n) \tag{15}$$

Then, type θ_n has an incentive to mimic type θ', a contradiction.

(ii) Similar to the case (i). By (A7), $C(\mu(\theta'), \theta') > C(\bar{\mu}, \theta')$. That is, for sufficiently large n, $C(\mu(\theta'), \theta') > C(\mu(\theta_n), \theta')$. Also for sufficiently large n, $u^S(a^R(\theta'), \theta', x) - u^S(a^R(\theta_n), \theta', x)$ is arbitrarily close to 0. Then,

$$u^S(a^R(\theta'), \theta', x) - kC(\mu(\theta'), \theta') < u^S(a^R(\theta_n), \theta', x) - kC(\mu(\theta_n), \theta'). \tag{16}$$

That is, type θ' has an incentive to mimic type θ_n, a contradiction.

Therefore, $\mu(\cdot)$ should be continuous on T. ■

Lemma 7 *Suppose $\mu(\cdot)$ is an equilibrium strategy and one-to-one on the positive measured subset $T \subseteq \Theta$. Then:*

$$\lim_{\theta'' \to \theta'} \frac{\mu(\theta'') - \mu(\theta')}{\theta'' - \theta'} = -\frac{U_1^S(a^R(\theta'), \theta', \mu(\theta'), x)a_1^R(\theta')}{U_3^S(a^R(\theta'), \theta', \mu(\theta'), x)}, \tag{17}$$

where θ', θ'' are taken from the interior of T.

Proof. The proof is almost same to Mailath (1987, Proposition 2). Fix $\theta' \in T$ and x. Let

$$\begin{aligned} U(\hat{\theta}, \theta, m) &\equiv U^S(a^R(\hat{\theta}), \theta, m, x), \\ g(\sigma_1, \sigma_2, \sigma_3) &\equiv U(\sigma_1, \sigma_2, \sigma_3) - U(\theta', \sigma_2, \mu(\theta')). \end{aligned}$$

Take $\theta'' \in T$ with $\theta'' \neq \theta'$ and define followings;

$$\begin{aligned} \sigma_1(\lambda) &\equiv \lambda\theta'' + (1-\lambda)\theta', \\ \sigma_2(\lambda) &\equiv \theta', \\ \sigma_3(\lambda) &\equiv \lambda\mu(\theta'') + (1-\lambda)\mu(\theta'), \\ \bar{\sigma}_1(\lambda) &\equiv \theta'', \\ \bar{\sigma}_2(\lambda) &\equiv \lambda\theta'' + (1-\lambda)\theta', \\ \bar{\sigma}_3(\lambda) &\equiv \mu(\theta''). \end{aligned}$$

Since $\mu(\cdot)$ is an equilibrium strategy and one-to-one on T, it should satisfy

$$\begin{aligned} & U(\theta', \theta', \mu(\theta')) \geq U(\theta'', \theta', \mu(\theta'')) \\ \iff\quad & g(\theta'', \theta', \mu(\theta'')) \leq 0. \qquad (18) \\ & U(\theta'', \theta'', \mu(\theta'')) \geq U(\theta', \theta'', \mu(\theta')) \\ \iff\quad & g(\theta'', \theta'', \mu(\theta'')) \geq 0. \qquad (19) \end{aligned}$$

First, expand $g(\theta'', \theta'', \mu(\theta''))$ around $(\theta'', \theta', \mu(\theta''))$. There exists $\lambda \in (0,1)$ and

$$g(\theta'', \theta'', \mu(\theta'')) = g(\theta'', \theta', \mu(\theta'')) + g_2(\theta'', \theta', \mu(\theta''))(\theta'' - \theta') + \frac{1}{2}g_{22}(\bar{\sigma}(\lambda))(\theta'' - \theta')^2.$$

Then expand $g_2(\theta'', \theta', \mu(\theta''))$ around $(\theta', \theta', \mu(\theta'))$. For some $\eta \in (0,1)$,

$$g_2(\theta'', \theta', \mu(\theta'')) = g_2(\theta', \theta', \mu(\theta')) + g_{12}(\sigma(\eta))(\theta'' - \theta') + g_{23}(\sigma(\eta))(\mu(\theta'') - \mu(\theta')).$$

Hence,

$$\begin{aligned} g(\theta'', \theta'', \mu(\theta'')) &= g(\theta'', \theta', \mu(\theta'')) + g_2(\theta', \theta', \mu(\theta'))(\theta'' - \theta') + g_{12}(\sigma(\eta))(\theta'' - \theta')^2 \\ &\quad + g_{23}(\sigma(\eta))(\mu(\theta'') - \mu(\theta'))(\theta'' - \theta') + \frac{1}{2}g_{22}(\bar{\sigma}(\lambda))(\theta'' - \theta')^2. \end{aligned}$$

Since $g_2(\sigma_1, \sigma_2, \sigma_3) = U_2(\sigma_1, \sigma_2, \sigma_3) - U_2(\theta', \sigma_2, \mu(\theta'))$, $g_2(\theta', \theta', \mu(\theta')) = 0$. Hence,

$$\begin{aligned} g(\theta'', \theta'', \mu(\theta'')) &= g(\theta'', \theta', \mu(\theta'')) + g_{12}(\sigma(\eta))(\theta'' - \theta')^2 + g_{23}(\sigma(\eta))(\mu(\theta'') - \mu(\theta'))(\theta'' - \theta') \\ &\quad + \frac{1}{2}g_{22}(\bar{\sigma}(\lambda))(\theta'' - \theta')^2. \end{aligned}$$

Equivalently,

$$
\begin{aligned}
g(\theta'',\theta',\mu(\theta'')) &= g(\theta'',\theta'',\mu(\theta'')) - g_{12}(\sigma(\eta))(\theta''-\theta')^2 - g_{23}(\sigma(\eta))(\mu(\theta'')-\mu(\theta'))(\theta''-\theta') \\
&\quad -\frac{1}{2}g_{22}(\bar{\sigma}(\lambda))(\theta''-\theta')^2. \qquad (20)
\end{aligned}
$$

By (18), (19), and (20),

$$
\begin{aligned}
0 &\geq g(\theta'',\theta',\mu(\theta'')) \\
&= g(\theta'',\theta'',\mu(\theta'')) - g_{12}(\sigma(\eta))(\theta''-\theta')^2 - g_{23}(\sigma(\eta))(\mu(\theta'')-\mu(\theta'))(\theta''-\theta') - \frac{1}{2}g_{22}(\bar{\sigma}(\lambda))(\theta''-\theta')^2 \\
&\geq -\Big[g_{12}(\sigma(\eta)) + \frac{1}{2}g_{22}(\bar{\sigma}(\lambda))\Big](\theta''-\theta')^2 - g_{23}(\sigma(\eta))(\mu(\theta'')-\mu(\theta'))(\theta''-\theta'). \qquad (21)
\end{aligned}
$$

Next, expand $g(\theta'',\theta',\mu(\theta''))$ around $(\theta',\theta',\mu(\theta'))$. For some $\gamma \in (0,1)$,

$$
\begin{aligned}
g(\theta'',\theta',\mu(\theta'')) &= g_1(\theta',\theta',\mu(\theta'))(\theta''-\theta') + g_3(\theta',\theta',\mu(\theta'))(\mu(\theta'')-\mu(\theta')) \\
&+ \frac{1}{2}\Big[g_{11}(\sigma(\gamma))(\theta''-\theta')^2 + 2g_{13}(\sigma(\gamma))(\theta''-\theta')(\mu(\theta'')-\mu(\theta')) + g_{33}(\sigma(\gamma))(\mu(\theta'')-\mu(\theta'))^2\Big]
\end{aligned}
$$

Divide both sides by $\theta''-\theta' > 0$, [16]

$$
\begin{aligned}
\frac{g(\theta'',\theta',\mu(\theta''))}{\theta''-\theta'} &= g_1(\theta',\theta',\mu(\theta')) + \frac{\mu(\theta'')-\mu(\theta')}{\theta''-\theta'}\Big\{g_3(\theta',\theta',\mu(\theta')) + g_{13}(\sigma(\gamma))(\theta''-\theta') \\
&\quad +\frac{1}{2}g_{33}(\sigma(\gamma))(\mu(\theta'')-\mu(\theta'))\Big\} + \frac{1}{2}g_{11}(\sigma(\gamma))(\theta''-\theta'). \qquad (22)
\end{aligned}
$$

Divide (21) by $\theta''-\theta' > 0$,

$$
0 \geq \frac{g(\theta'',\theta',\mu(\theta''))}{\theta''-\theta'} \geq -\Big[g_{12}(\sigma(\eta)) + \frac{1}{2}g_{22}(\bar{\sigma}(\eta))\Big](\theta''-\theta') - g_{23}(\sigma(\eta))(\mu(\theta'')-\mu(\theta')). \qquad (23)
$$

By (22) and (23),

$$
\begin{aligned}
0 &\geq g_1(\theta',\theta',\mu(\theta')) + \frac{\mu(\theta'')-\mu(\theta')}{\theta''-\theta'}\Big\{g_3(\theta',\theta',\mu(\theta')) + g_{13}(\sigma(\gamma))(\theta''-\theta') \\
&\quad +\frac{1}{2}g_{33}(\sigma(\gamma))(\mu(\theta'')-\mu(\theta'))\Big\} + \frac{1}{2}g_{11}(\sigma(\gamma))(\theta''-\theta') \\
&\geq -\Big[g_{12}(\sigma(\eta)) + \frac{1}{2}g_{22}(\bar{\sigma}(\eta))\Big](\theta''-\theta') - g_{23}(\sigma(\eta))(\mu(\theta'')-\mu(\theta')). \qquad (24)
\end{aligned}
$$

By (A1), $|U_{ij}| < \infty$ for any i, j. Then it implies $|g_{ij}| < \infty$. By Lemma 6, $\mu(\cdot)$ is continuous on T. Then $\mu(\theta'') \to \mu(\theta')$ as $\theta'' \to \theta'$. Take the limit of (24) as $\theta'' \to \theta'$,

$$
0 \geq g_1(\theta',\theta',\theta') + \lim_{\theta'' \to \theta'} \frac{\mu(\theta'')-\mu(\theta')}{\theta''-\theta'} g_3(\theta',\theta',\mu(\theta')) \geq 0.
$$

Then,

$$
U_1(\theta',\theta',\mu(\theta')) + \lim_{\theta'' \to \theta'} \frac{\mu(\theta'')-\mu(\theta')}{\theta''-\theta'} U_3(\theta',\theta',\mu(\theta')) = 0. \qquad (25)
$$

[16] If $\theta''-\theta' < 0$, then all inequalities are converse.

By (A7), $U_3 \neq 0$. Therefore,

$$\lim_{\theta'' \to \theta'} \frac{\mu(\theta'') - \mu(\theta')}{\theta'' - \theta'} = -\frac{U_1^S(a^R(\theta'), \theta', \mu(\theta'), x) a_1^R(\theta')}{U_3^S(a^R(\theta'), \theta', \mu(\theta'), x)}. \blacksquare$$

Proof of Theorem 5. Set $T = \Theta$. By Lemma 6, $\mu(\cdot)$ is continuous on $[0, 1]$. By Lemma 7, the fully separating strategy $\mu(\cdot)$ should be differentiable on $(0, 1)$ and then the differential equation (4) is obtained. ■

Proof of Proposition 1

Suppose to contradiction that there exists a fully separating equilibrium. By Theorem 5, the equilibrium strategy $\mu(\cdot)$ is differentiable on $(0, 1)$ and where it is differentiable, it satisfies (4). Since $x = 0$ and (A5), $u_1^S(a^R(\theta), \theta, 0)a_1^R(\theta) = 0$; that is, $\mu_1(\cdot) = 0$ where it is differentiable. However, this contradicts that $\mu(\cdot)$ is continuous and one-to-one. Therefore, there exists no fully separating equilibrium. ■

Proof of Corollary 1

Suppose to the contrary that there exists a semi-separating equilibrium. Let $T \subset \Theta$ be a positive measured information revealing set. Then, as an analogous to Proposition 1, we can show that the equilibrium strategy $\mu(\cdot)$ should be continuous and differentiable on the interior of T, and satisfy (4). However, $\mu_1(\cdot) = 0$ on the interior of T, a contradiction. ■

Proof of Lemma 2

(Necessity) Suppose, by way of contradiction, that $\mu(\theta) \neq \mu(\theta')$. Without loss of generality, assume that $\mu(\theta) < \mu(\theta')$. Then, type θ' has no incentive to send a message $\mu(\theta')$ on the equilibrium path because $C(\mu(\theta), \theta') < C(\mu(\theta'), \theta')$ and $\alpha(\mu(\theta)) = \alpha(\mu(\theta'))$, a contradiction to $\mu(\theta')$ being sent on the equilibrium path.

(Sufficiency) It is trivial. Because $\mu(\theta) = \mu(\theta')$ and the receiver's strategy depends only on the message sent by the sender, $\alpha(\mu(\theta)) = \alpha(\mu(\theta'))$ on the equilibrium path; that is, the both types induce the same action. ■

Proof of Lemma 3

This proof is almost same to Kartik (2005a, Lemma 1). Suppose, by way of contradiction, that there exist types $\theta, \theta', \theta'' \in \Theta$ with $\theta < \theta' < \theta''$ such that types θ and θ'' induce the action a_1, whereas θ' induces the action $a_2 \neq a_1$. By Lemma 2, $\mu(\theta) = \mu(\theta'') \neq \mu(\theta')$. However, by the monotonicity of the signaling strategy, $\mu(\theta) \leq \mu(\theta')$ and $\mu(\theta') \leq \mu(\theta'')$, a contradiction. ■

Proof of Lemma 4

By Lemma 3, the set of types that induces a same action is an interval. As long as the equilibrium is not the trivial pooling equilibrium, there exist at least two different equilibrium messages $m_{i-1} < m_i$ such that $m_{i-1} = \mu(\theta), \forall \theta \in (\theta_{i-1}, \theta_i)$ and $m_i = \mu(\theta'), \forall \theta' \in (\theta_i, \theta_{i+1})$, and $\alpha(m_{i-1}) \neq \alpha(m_i)$ by Lemma 2. Then, on the equilibrium path, the receiver can learn the interval, to which the type belongs by observing the message. Hence $\alpha(m_i) = \bar{a}(\theta_i, \theta_{i+1})$. Suppose, by way of contradiction, that (6) does not hold; that is, there exists i such that either one of the followings holds;

$$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) - kC(m_{i-1}, \theta_i) \quad > \quad u^S(\bar{a}(m_i, \theta_{i+1}), \theta_i, 0) - kC(m_i, \theta_i), \tag{26}$$

$$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) - kC(m_{i-1}, \theta_i) \quad < \quad u^S(\bar{a}(m_i, \theta_{i+1}), \theta_i, 0) - kC(m_i, \theta_i). \tag{27}$$

Case (i): (26) holds Since $u^S(a, \theta, 0) - kC(m, \theta)$ is continuous in θ, there exists $\delta > 0$ such that $\forall \theta' \in (\theta_i, \theta_i + \delta)$, $u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta', 0) - kC(m_{i-1}, \theta') > u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta', 0) - kC(m_i, \theta')$, which contradicts with type $\theta' \in (\theta_i, \theta_{i+1})$ sending message m_i to induce action $\bar{a}(\theta_i, \theta_{i+1})$ on the equilibrium path.

Case (ii): (27) holds Similar to the above, there exists $\eta > 0$ such that $\forall \theta'' \in (\theta_i - \eta, \theta_i) \subset (\theta_{i-1}, \theta_i)$ has an incentive to send message m_i to induce action $\bar{a}(\theta_{i-1}, \theta_i)$, a contradiction.

Then, on the monotone equilibrium, (6) should hold. ■

Proof of Proposition 2

Suppose that there exists a monotone partition equilibrium with N segments. Let $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ be an equilibrium partition and $\{m_0, m_1, \ldots\}$ be a monotone equilibrium message profile. By Lemma 4, each boundary type should satisfy the boundary conditions (7); that is, for $i = 1, 2, \ldots$,

$$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) - kC(m_{i-1}, \theta_i) = u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta_i, 0) - kC(m_i, \theta_i). \tag{28}$$

By the construction, for $i = 1, 2, \ldots$,

$$\bar{a}(\theta_{i-1}, \theta_i) < a^R(\theta_i) < \bar{a}(\theta_i, \theta_{i+1}). \tag{29}$$

Since $m_i > m_{i-1}$, $C(m_i, \theta_i) > C(m_{i-1}, \theta_i)$. Then, by (28), $u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta_i, 0) < u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta_i, 0)$. Since u^S is continuous in θ, there exists $\hat{\theta}_i \in [0, 1]$ such that $u^S(\bar{a}(\theta_{i-1}, \theta_i), \hat{\theta}_i, 0) = u^S(\bar{a}(\theta_i, \theta_{i+1}), \hat{\theta}_i, 0)$. Therefore, for $i = 1, 2, \ldots$, $\hat{\theta}_i < a^R(\theta_i)$, and

$$\bar{a}(\theta_{i-1}, \theta_i) < \hat{\theta}_i < \bar{a}(\theta_i, \theta_{i+1}). \tag{30}$$

By (29) and (30), for $i = 1, 2, \ldots$,

$$\bar{a}(\theta_i, \theta_{i+1}) - \bar{a}(\theta_{i-1}, \theta_i) > a^R(\theta_i) - \hat{\theta}_i > 0. \tag{31}$$

Let define the function $\hat{a}^S : [\theta_1, 1] \to [0, 1]$ by

$$\hat{a}^S(\theta) = \begin{cases} \hat{\theta}_i & \text{if } \theta \in \{\theta_1, \theta_2, \ldots\} \\ \dfrac{\hat{\theta}_{i+1} - \hat{\theta}_i}{\theta_{i+1} - \theta_i}\theta + \dfrac{\hat{\theta}_i \theta_{i+1} - \theta_i \hat{\theta}_{i+1}}{\theta_{i+1} - \theta_i} & \text{if } \theta \in (\theta_i, \theta_{i+1}) \text{ for } i = 1, 2, \ldots \end{cases}$$

Since $\hat{\theta}_{i+1} > \theta_i$ for all i by (30), $\hat{a}^S$ is continuous and strictly increasing. Moreover, $a_1^R(\cdot) > 0$ and $a^R(\theta_i) > \hat{a}^S(\theta_i)$for all i. Thus, $a^R(\theta) - \hat{a}^S(\theta) > 0, \forall \theta \in [\theta_1, 1]$. Since $a^R(\theta) - \hat{a}^S(\theta)$ is continuous on $[\theta_1, 1]$, by the Weierstrass Theorem, there exists $\tilde{\theta} \in [\theta_1, 1]$ such that

$$a^R(\theta) - \hat{a}^S(\theta) \geq a^R(\tilde{\theta}) - \hat{a}^S(\tilde{\theta}) \equiv \epsilon > 0, \forall \theta \in [\theta_1, 1].$$

Therefore, by (31), for $i = 1, 2, \ldots$,

$$\begin{aligned} \bar{a}(\theta_i, \theta_{i+1}) - \bar{a}(\theta_{i-1}, \theta_i) &> a^R(\theta_i) - \hat{\theta}_i \\ &= a^R(\theta_i) - \hat{a}^S(\theta_i) > \epsilon. \end{aligned}$$

Since the set of actions induced in the equilibrium is $[a^R(0), a^R(1)]$. That is, the number of induced actions should be finite. In other words, N should be, at most, finite in any monotone partition equilibrium in the common interest almost-cheap talk game. ■

Proof of Proposition 3

First, we show one additional lemma;

Lemma 8 *Suppose (B1), and there exist a partition* $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ *and a profile of messages* $\{m_0, \ldots, m_{N-1}\}$ *with* $m_i < m_{i+1}$ *for* $i = 0, \ldots, N-2$ *satisfying (a). Consider the action-message pairs* $\{(\bar{a}(\theta_0, \theta_1), m_0), \ldots, (\bar{a}(\theta_{N-1}, \theta_N), m_{N-1})\}$*, and call* $(\bar{a}(\theta_i, \theta_{i+1}), m_i)$ *the ith pair. Then all types belong to the interval* (θ_i, θ_{i+1}) *prefers the ith pair the most, and less prefers the jth pair as j becomes further away from i.*

Proof of Lemma 8. For $i = 1, \ldots, N-1$, let

$$h(\theta; \theta_{i-1}, \theta_i, \theta_{i+1}) \equiv u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta, 0) - k(C(m_i, \theta) - C(m_{i-1}, \theta)).$$

Then

$$\frac{\partial h}{\partial \theta} = u_2^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - u_2^S(\bar{a}(\theta_{i-1}, \theta_i), \theta, 0) - k(C_2(m_i, \theta) - C_2(m_{i-1}, \theta)).$$

By the definition of $\bar{a}(\cdot, \cdot)$, $\bar{a}(\theta_i, \theta_{i+1}) > \bar{a}(\theta_{i-1}, \theta_i)$. Because $u_{12}^S > 0$, $u_2^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - u_2^S(\bar{a}(\theta_{i-1}, \theta_i), \theta, 0) > 0, \forall \theta \in \Theta$. Also, because $m_{i-1} < m_i$ and $C_{12} \leq 0$, $C_2(m_i, \theta) - C_2(m_{i-1}, \theta) \leq 0, \forall \theta \in \Theta$. Therefore, for all $\theta \in \Theta$, $\partial h / \partial \theta > 0$. By the hypothesis, $h(\theta_i; \theta_{i-1}, \theta_i, \theta_{i+1}) = 0$ for $i = 1, \ldots, N-1$. Then for all $\theta > \theta_i$, $h(\theta; \theta_{i-1}, \theta_i, \theta_{i+1}) > 0$; that is,

$$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta, 0) - kC(m_{i-1}, \theta) < u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - kC(m_i, \theta).$$

Also, for all $\theta < \theta_i$, $h(\theta; \theta_{i-1}, \theta_i, \theta_{i+1}) < 0$; that is,

$$u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta, 0) - kC(m_{i-1}, \theta) > u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - kC(m_i, \theta).$$

Therefore, for all $\theta \in (\theta_i, \theta_{i+1})$,

$$\begin{aligned} u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - kC(m_i, \theta) &> u^S(\bar{a}(\theta_{i-1}, \theta_i), \theta, 0) - kC(m_{i-1}, \theta) \\ &> u^S(\bar{a}(\theta_{i-2}, \theta_{i-1}), \theta, 0) - kC(m_{i-2}, \theta) \\ &> \cdots \\ &> u^S(\bar{a}(\theta_0, \theta_1), \theta, 0) - kC(m_0, \theta), \end{aligned}$$

and

$$\begin{aligned} u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - kC(m_i, \theta) &> u^S(\bar{a}(\theta_{i+1}, \theta_{i+2}), \theta, 0) - kC(m_{i+1}, \theta) \\ &> u^S(\bar{a}(\theta_{i+2}, \theta_{i+3}), \theta, 0) - kC(m_{i+2}, \theta) \\ &> \cdots \\ &> u^S(\bar{a}(\theta_{N-1}, \theta_N), \theta, 0) - kC(m_{N-1}, \theta). \end{aligned}$$

Thus, all types belongs to the interval (θ_i, θ_{i+1}) prefers the ith pair most, and less prefers the jth pair as j is far away from i. ■

Proof of Proposition 3.

1. Suppose that there exists a monotone equilibrium. By Proposition 1 and Corollary 1, at most countably many actions are induced on the equilibrium path, and by Lemma 3, the set of types who induce the same action is an interval. Then, the results are straightforward from Lemma 4.

2. Suppose that there exist a partition $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ and a profile of messages $\{m_0 \equiv 0, \ldots, m_{N-1}\}$ with $m_i < m_{i+1}$ for $i = 0, \ldots, N-2$ satisfying (a). Show that the following is a PBE; [17]

$$\begin{aligned}
\mu(\theta) &= m_i \text{ if } \theta \in [\theta_i, \theta_{i+1}) \text{ for all } i = 0, \ldots, N-1, \\
\alpha(m) &= \begin{cases} \bar{a}(\theta_i, \theta_{i+1}) & \text{if } m \in [m_i, m_{i+1}) \text{ for all } i = 0, \ldots, N-2 \\ \bar{a}(\theta_{N-1}, \theta_N) & \text{if } m \in [m_{N-1}, 1], \end{cases} \\
\mathcal{P}(\theta|m) &= \begin{cases} \dfrac{f(\theta)}{\int_{\theta_i}^{\theta_{i+1}} f(\tilde{\theta}) d\tilde{\theta}} & \text{if } m \in [m_i, m_{i+1}) \text{ and } \theta \in [\theta_i, \theta_{i+1}) \text{ for all } i = 0, \ldots, N-2 \\ \dfrac{f(\theta)}{\int_{\theta_{N-1}}^{\theta_N} f(\tilde{\theta}) d\tilde{\theta}} & \text{if } m \in [m_{N-1}, 1] \text{ and } \theta \in [\theta_{N-1}, \theta_N] \\ 0 & \text{Otherwise} \end{cases}
\end{aligned}$$

Given the receiver's belief $\mathcal{P}(\cdot|\cdot)$, $\alpha(\cdot)$ is her best response. Next, check the optimality of $\mu(\cdot)$ given the receiver's best response $\alpha(\cdot)$. Fix $\theta \in (\theta_i, \theta_{i+1})$. If he follows this strategy, then action $\bar{a}(\theta_i, \theta_{i+1})$ is induced with paying cost $kC(m_i, \theta)$. By Lemma 8, the most preferred action-message pair among $\{(\bar{a}(\theta_0, \theta_1), m_0), \ldots, (\bar{a}(\theta_{N-1}, \theta_N), m_{N-1})\}$ is $(\bar{a}(\theta_i, \theta_{i+1}), m_i)$. Then he has no incentive to mimic types in the other segments. If he sends a message $m \in (m_i, m_{i+1})$, then $\bar{a}(\theta_i, \theta_{i+1})$ is induced. However, he has to pay more cost than m_i because $C(m, \theta) > C(m_i, \theta)$ by (A7). Thus, he has no incentive to send this kind of messages. Also, if he sends a message $m \in (m_j, m_{j+1})$ for $j > i$, then $\bar{a}(\theta_j, \theta_{j+1})$ is induced. However,

$$\begin{aligned}
u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - kC(m_i, \theta) &> u^S(\bar{a}(\theta_j, \theta_{j+1}), \theta, 0) - kC(m_j, \theta) \\
&> u^S(\bar{a}(\theta_j, \theta_{j+1}), \theta, 0) - kC(m, \theta).
\end{aligned}$$

Thus, he has no incentive to send this kind of messages. Similarly, if he sends a message $m \in (m_h, m_{h+1})$ for $h < i$, then $\bar{a}(\theta_h, \theta_{h+1})$ is induced. However,

$$\begin{aligned}
u^S(\bar{a}(\theta_i, \theta_{i+1}), \theta, 0) - kC(m_i, \theta) &> u^S(\bar{a}(\theta_h, \theta_{h+1}), \theta, 0) - kC(m_h, \theta) \\
&> u^S(\bar{a}(\theta_h, \theta_{h+1}), \theta, 0) - kC(m, \theta).
\end{aligned}$$

Then he has no incentive to send this kind of messages. Next, suppose the sender is the boundary type θ_i. By the boundary condition (7), he is indifferent from inducing $\bar{a}(\theta_i, \theta_{i+1})$ by paying cost $kC(m_i, \theta)$ and inducing $\bar{a}(\theta_{i-1}, \theta_i)$ by paying cost $kC(m_{i-1}, \theta)$. As an analogue

[17] Note that on the equilibrium, if the receiver observes an off the equilibrium path message $m \in (m_i, m_{i+1})$ for $i = 0, \ldots, N-2$, then her belief is identical to that of which she observes message m_i; that is, $\mathcal{P}(\cdot|m) = \mathcal{P}(\cdot|m_i), \forall m \in (m_i, m_{i+1})$. Similarly, $\mathcal{P}(\cdot|m) = \mathcal{P}(\cdot|m_{N-1}), \forall m > m_{N-1}$.

to the above, by Lemma 8, he has no incentive to mimic types in other segments and, moreover, also no incentive to send any off the equilibrium path messages. Thus, sending m_i to induce action $\bar{a}(\theta_i, \theta_{i+1})$ is a best response. Therefore, the sender never deviates from $\mu(\cdot)$; that is, it is his best response. Finally, given $\mu(\cdot)$, the belief $\mathcal{P}(\cdot|\cdot)$ is consistent with Bayes' rule on the equilibrium path. Thus, these construct a PBE. ■

Proof of Proposition 4

Suppose that there exists a partition equilibrium with N segments in the pure cheap talk game, and let $< \theta_0^* \equiv 0, \theta_1^*, \ldots, \theta_N^* \equiv 1 >$ is the equilibrium partition. Since each boundary type should satisfy (9) on the equilibrium, $W_i(\theta_1^*, \theta_2^*, \ldots, \theta_{N-1}^*; 0) = 0$ for $i = 1, \ldots, N-1$. By the hypothesis, the determinant of the Jacobian J at $(\theta_1^*, \theta_2^*, \ldots, \theta_{N-1}^*; 0)$ is not zero. Then, by the Implicit Function Theorem, for sufficiently small $k > 0$, there exist $(\theta_1, \theta_2, \ldots, \theta_{N-1})$ solving the system of equations, $W_i(\theta_1, \theta_2, \ldots, \theta_{N-1}; k) = 0$ for $i = 1, \ldots, N-1$, on the neighborhood of $(\theta_1^*, \theta_2^*, \ldots, \theta_{N-1}^*)$, and then $\theta_i = \psi_i(k)$ for $i = 1, \ldots, N-1$. That is, for sufficiently small $k > 0$, there exists a partition $< \theta_0 \equiv 0, \theta_1, \ldots, \theta_N \equiv 1 >$ and a monotone message profile $\{m_0, \ldots, m_{N-1}\}$ with $m_i = \theta_i$ for $i = 0, \ldots, N-1$ satisfying (7). ■

Proof of Corollary 2

Lemma 9 *Consider the uniform-quadratic model. Then for any natural number N, there exists a partition equilibrium with N segments in the common interest pure cheap talk game.*

Proof of Lemma 9 Fix $N \in \mathbb{N}$ arbitrarily. Then, for $j = 1, \ldots, N-1$, the following arbitrage condition holds;

$$\begin{aligned} -\Big(\frac{\theta_{j-1} + \theta_j}{2} - \theta_j\Big)^2 &= -\Big(\frac{\theta_j + \theta_{j+1}}{2} - \theta_j\Big)^2 \\ \Longleftrightarrow \theta_{j+1} - \theta_j &= \theta_1 - \theta_0 = \theta_1. \end{aligned} \tag{32}$$

That is, by dividing the entire type space into N intervals with same length, we can construct a partition satisfying the arbitrage conditions. Therefore, there exists a partition equilibrium with N segments. Since N is arbitrary, for every natural number N there exists a partition equilibrium with N segments. ■

Lemma 10 *In the uniform-quadratic model, for any $N \geq 2$, the determinant of J at the boundaries of N steps partition equilibrium in the common interest pure cheap talk game, $(\theta_1^*, \theta_2^*, \ldots, \theta_{N-1}^*)$, is not zero.*

Proof of Lemma 10

$$
\begin{aligned}
J &= \frac{1}{2}\begin{pmatrix}
-\theta_2^* & -\theta_1^*+\theta_2^* & 0 & 0 & \dots & 0 & 0 & 0 \\
-\theta_1^*+\theta_2^* & \theta_1^*-\theta_3^* & -\theta_2^*+\theta_3^* & 0 & \dots & 0 & 0 & 0 \\
0 & -\theta_2^*+\theta_3^* & \theta_2^*-\theta_4^* & -\theta_3^*+\theta_4^* & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & -\theta_{N-2}^*+\theta_{N-1}^* & \theta_{N-2}^*-1
\end{pmatrix} \\
&= \frac{1}{2N}\begin{pmatrix}
-2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & -2
\end{pmatrix}.
\end{aligned} \tag{33}
$$

Then, the determinant of the Jacobian $|J|$ is

$$|J| = \frac{(-1)^{N-1}N}{(2N)^{N-1}} \neq 0. \quad \blacksquare \tag{34}$$

Proof of Corollary 2

By Lemmas 9 and 10, for any $N \in \mathbb{N}$, there exists a partition equilibrium with N segments in the common interest pure cheap talk game, and the determinant of the Jacobian is not zero. Thus, by Proposition 4, for sufficiently small $k > 0$, there exist a partition with N segments and a monotone message profile satisfying (7). Thus, By Proposition 3, there exists a partition equilibrium with N segments in the common interest almost-cheap talk game. $\blacksquare$

Proof of Lemma 5

This proof is almost same to the proof of Kartik (2005a, Lemma 1) and Kartik (2008, Lemma A.1). Let

$$\mu_1 = g(\mu,\theta) = \frac{u_1^S(a^R(\theta),\theta,x)a_1^R(\theta)}{kC_1(\mu,\theta)}, \mu(0) = 0. \tag{35}$$

First, show that there exists a unique solution $\tilde{\mu}$ to (35) on the neighborhood of $\theta = 0$. Let $\mathcal{C}([a,b))$ be the set of all real-valued functions on $[a,b)$ that have a continuous derivative at all $t \in (a,b)$ and, in addition, have a right-continuous derivative at a that is continuous from the right at a. $\mathcal{C}([a,b])$ can be defined similarly. Note that g is continuous on $M \times \Theta = [0,1]^2$ and $g \in \mathcal{C}([0,1]^2)$ by (A1) and (A7), and then g satisfies the Lipschitz condition in μ on $[0,1]^2$.[18] Thus, by the Picard

[18]See de la Fuente (2000).

Local Existence Theorem, there exists a unique solution $\tilde{\mu}$ to (35) on $[0,\delta)$ for some $\delta > 0$. That is, $\tilde{\mu} \in \mathcal{C}([0,\delta))$ and $\tilde{\mu}_1 > 0$ on the range.

Next show that there exists a unique extension of $\tilde{\mu}$ on $[0,\delta+\beta)$. Since $\tilde{\mu}_1 > 0$ on $(0,\delta)$, there exists $\bar{\mu} \equiv \lim_{\theta\to\delta} \tilde{\mu}(\theta)$, and suppose that $\bar{\mu} < 1$. Now, consider the following problem;

$$\nu_1 = g(\nu,\theta) = \frac{u_1^S(a_1^R(\theta),\theta,x)a_1^R(\theta)}{kC_1(\nu,\theta)}, \nu(\delta) = \bar{\mu}. \tag{36}$$

Similarly, g satisfies the Lipschitz condition in ν on $[\bar{\mu},1] \times [\delta,1]$. Then by the Picard Local Existence Theorem, there exists a unique solution $\tilde{\nu}$ to (36) on $[\delta,\delta+\beta)$ for some $\beta > 0$. Note that $\tilde{\nu} \in \mathcal{C}([\delta,\delta+\beta))$ and $\tilde{\nu}_1 > 0$ on the range. Hence, define $\hat{\mu}$ by

$$\hat{\mu}(\theta) = \begin{cases} \tilde{\mu}(\theta) & \text{if } \theta \in [0,\delta) \\ \bar{\mu} & \text{if } \theta = \delta \\ \tilde{\nu}(\theta) & \text{if } \theta \in (\delta,\delta+\beta) \end{cases}$$

It remains to show that $\hat{\mu} \in \mathcal{C}([0,\delta+\beta)$.

Claim 2 $\hat{\mu}(\theta) = \int_0^\theta g(\hat{\mu}(s),s)ds, \forall s \in [0,\delta+\beta)$.

Proof of Claim 2 If $\theta \in [0,\delta)$, then it is obvious. If $\theta = \delta$, then

$$\begin{aligned} \hat{\mu}(\delta) &= \bar{\mu} = \lim_{\theta\to\delta-} \tilde{\mu}(\theta) = \lim_{\theta\to\delta-} \hat{\mu}(\theta) \\ &= \lim_{\theta\to\delta-} \int_0^\theta g(\hat{\mu}(s),s)ds \\ &= \int_0^\delta g(\hat{\mu}(s),s)ds. \end{aligned} \tag{37}$$

If $\theta > \delta$, then

$$\begin{aligned} \hat{\mu}(\theta) &= \tilde{\nu}(\theta) = \bar{\mu} + \int_\delta^\theta g(\tilde{\nu}(s),s)ds \\ &= \bar{\mu} + \int_\delta^\theta g(\hat{\mu}(s),s)ds. \end{aligned} \tag{38}$$

Combining (37) and (38), we obtain that $\hat{\mu}(\theta) = \int_0^\theta g(\hat{\mu}(s),s)ds$. □
By definition, $\hat{\mu}$ is continuous on $[0,\delta+\beta)$. Then $g(\hat{\mu}(\theta),\theta)$ is also continuous on $[0,\delta+\beta)$. By Claim 2, take the derivative at $\theta = \delta$, and then $\hat{\mu}_1(\delta) = g(\hat{\mu}(\delta),\delta)$. Thus, $\hat{\mu}_1$ can be defined on $[0,\delta+\beta)$ and it is continuous on this range.[19] That is, $\hat{\mu}$ is continuously differentiable and $\hat{\mu}_1 > 0$.

We can continue this procedure until $\bar{\mu} = 1$. If $\bar{\mu} = 1$, that is, $\lim_{\theta\to\delta-} \tilde{\mu}(\theta) = 1$, then set $\bar{\theta} = \delta$. Therefore, if $\bar{\theta} < 1$, then $\tilde{\mu}(\bar{\theta}) = 1$ should hold; otherwise, we can still continue this extension procedure. □

[19] At $\theta = 0$, there exists a right-derivative and right-continuous at the point.

Proof of Proposition 5

Fix $k > 0$ arbitrarily. The proof is constructed by the following steps.

Lemma 11 *(Based on Kartik (2008), Lemma A.3) Fix $k > 0$ and $x_1 > x_2 > 0$. Then for all $\theta \in (0, \min\{\bar{\theta}(x_1), \bar{\theta}(x_2)\}]$, $\mu(\theta; x_1) > \mu(\theta; x_2)$*

Proof of Lemma 11 The proof is almost same to Kartik (2008, Lemma A.3) with trivial change. Fix $k > 0$ and $x_1 > x_2 > 0$. Then $0 < \mu_1(\cdot|x_1), \mu_1(\cdot|x_2) < \infty$. That is, $\bar{\theta}(x_1), \bar{\theta}(x_2) > 0$, and then each domain of $\mu(\cdot|\cdot)$ is well-defined. Suppose to the contradiction that $\mu(\hat{\theta}|x_2) \geq \mu(\hat{\theta}|x_1)$ for some $\hat{\theta} \in (0, \min\{\bar{\theta}(x_1), \bar{\theta}(x_2)\}]$. Since $x_1 > x_2 > 0$ and $u_{13}^S > 0$, for all $\theta \in \Theta$, $u_1^S(a^R(\theta), \theta, x_1)a_1^R(\theta) > u_1^S(a^R(\theta), \theta, x_2)a_1^R(\theta)$. For any $\theta \in \Theta$, if $\mu(\theta|x_2) \geq \mu(\theta|x_1)$, then $C_1(\mu(\theta|x_1), \theta) \leq C_1(\mu(\theta|x_2), \theta)$. Hence, $\mu(\theta|x_2) \geq \mu(\theta|x_1)$ implies that $\mu_1(\theta|x_1) > \mu_1(\theta|x_2)$. By the hypothesis, $\mu_1(\hat{\theta}|x_1) > \mu_1(\hat{\theta}|x_2)$. These imply that for all $\theta \in (0, \hat{\theta}), \mu(\theta|x_2) > \mu(\theta|x_1)$. Then, for all $\theta \in (0, \hat{\theta}), \mu_1(\theta|x_1) > \mu_1(\theta|x_2)$. Representing them by inverse functions, we have that for all $m \in (0, \mu(\hat{\theta}|x_1))$,

$$\frac{d[(\mu(\cdot|x_1))^{-1}]}{dm} = \frac{1}{d\mu(\cdot|x_1)/d\theta} < \frac{1}{d\mu(\cdot|x_2)/d\theta} = \frac{d[(\mu(\cdot|x_2))^{-1}]}{dm}. \tag{39}$$

However, $(\mu(\cdot|x_1))^{-1}(0) = (\mu(\cdot|x_2))^{-1}(0) = 0$ and $(\mu(\cdot|x_1))^{-1}(\hat{m}) \geq (\mu(\cdot|x_2))^{-1}(\hat{m})$, where $\hat{m} = \mu(\hat{\theta}|x_1)$. Therefore, the conditions contradicts to (39). ■

Claim 3 *$\bar{\theta}(\cdot)$ is strictly decreasing in x if $\bar{\theta}(x) < 1$.*

Proof of Claim 3. Suppose that $\bar{\theta}(x) < 1$, and, by way of contradiction, that $\bar{\theta}(x') \leq \bar{\theta}(x)$ for $x' < x$. Then, by Lemma 11, $\forall \theta \in (0, \bar{\theta}(x')]$, $\mu(\theta; x) > \mu(\theta; x')$. That is, $\mu(\bar{\theta}(x'), x) > \mu(\bar{\theta}(x'), x')$. Since $\bar{\theta}(x) < 1$, $\bar{\theta}(x') < 1$. However, by Lemma 5-4, $\mu(\bar{\theta}(x'), x') = 1 < \mu(\bar{\theta}(x'), x)$, a contradiction. □

Claim 4 *If $\bar{\theta}(x) = 1$, then $\bar{\theta}(x'') = 1, \forall x'' \in [0, x]$.*

Proof of Claim 4. Suppose, by way of contradiction, that there exists $\tilde{x} \in [0, x)$ such that $\bar{\theta}(\tilde{x}) < 1$. By Lemma 11, $\forall \theta \in (0, \bar{\theta}(\tilde{x})]$, $\mu(\theta; x) > \mu(\theta; \tilde{x})$. Especially, by Lemma 5-4,

$$1 > \mu(\bar{\theta}(\tilde{x}); x) > \mu(\bar{\theta}(\tilde{x}); \tilde{x}) = 1,$$

a contradiction. □

First, suppose that $\bar{\theta}(x) = 1$. By Claim 4, $\forall x' \in (0, x], \bar{\theta}(x') = 1$. Then, we set $x(k) = x$. Next, suppose that $\bar{\theta}(x) < 1$. Let $X \equiv \{x > 0 | \bar{\theta}(x) = 1\}$.

Claim 5 *The set X is non-empty.*

Proof of Claim 5. Suppose, by way of contradiction, that $\forall x \in (0,1], \bar{\theta}(x) < 1$. By (A1) and (A7), C_1 is continuous on $[0,1]^2$ and $C_1 > 0$. Then, by the Weierstrass Theorem, there exists $(\underline{m}, \underline{\theta}) \in [0,1]^2$ such that $C_1(m,\theta) \geq C_1(\underline{m},\underline{\theta}) \equiv \underline{C} > 0, \forall (m,\theta) \in [0,1]^2$. Let

$$\hat{\mu}(\theta; x) \equiv \begin{cases} \mu(\theta; x) & \text{if } \theta \in [0, \bar{\theta}(x)] \\ 0 & \text{if } \theta \in (\bar{\theta}(x), 1] \end{cases}$$

Note that for all $x > 0$,

$$\begin{aligned} 1 = \int_0^{\bar{\theta}(x)} \frac{u_1^S(a^R(s), s, x) a_1^R(s)}{kC_1(\mu(s;x), s)} ds &< \int_0^1 \frac{u_1^S(a^R(s), s, x) a_1^R(s)}{kC_1(\hat{\mu}(s;x), s)} ds \\ &< \int_0^1 \frac{u_1^S(a^R(s), s, x) a_1^R(s)}{k\underline{C}} ds \\ &= \frac{1}{k\underline{C}} \int_0^1 u_1^S(a^R(s), s, x) a_1^R(s) ds. \end{aligned} \tag{40}$$

Show that $\int_0^1 u_1^S(a^R(s), s, x) a_1^R(s) ds$ is right-continuous at $x = 0$. Take a convergent sequence $x_n \to 0$ such that $x_n > x_{n+1}, \forall n \in \mathbb{N}$. Let

$$\begin{aligned} h_n(\theta) &\equiv u_1^S(a^R(\theta), \theta, x_n) a_1^R(\theta), \\ h(\theta) &\equiv u_1^S(a^R(\theta), \theta, 0) a_1^R(\theta). \end{aligned}$$

Since $u_{13} > 0$ and $a_1^R > 0$, $h_n > h_{n+1}, \forall n \in \mathbb{N}$. Also, note that each h_n function is a measurable function. Thus, by the Monotone Convergence Theorem,

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 h_n(s) ds &= \int_0^1 \lim_{n\to\infty} h_n(s) ds \\ &= \int_0^1 h(s) ds. \end{aligned}$$

The last equality is implied by the continuity of h_n in x. Therefore, it is right-continuous at $x = 0$. Moreover, since $u_{13} > 0$ and $a_1^R > 0$, $\int_0^1 u_1^S(a^R(s), s, x) a_1^R(s) ds$ is strictly increasing in x. Note that $\int_0^1 u_1^S(a^R(s), s, 0) a_1^R(s) ds = 0$ Thus, for sufficiently small $x' > 0$,

$$\frac{1}{k\underline{C}} \int_0^1 u_1^S(a^R(s), s, x') a_1^R(s) ds < 1. \tag{41}$$

Thus, (40) contradicts with (41). Therefore, X is non-empty. $\square$

By Claim 5, the set X is non-empty. Also, by Claim 3, $\forall x' \in X, x' < x$. Since the set X is non-empty and bounded above by x, there exists a supremum of X. Let the supremum describe $\bar{x}$. We have to consider the following two cases: (i) $\bar{x} \in X$ or (ii) $\bar{x} \notin X$.

Case (i): In this case $\bar{\theta}(\bar{x}) = 1$. By Claim 4, $\forall x'' \in (0, \bar{x}], \bar{\theta}(x'') = 1$. Then, we set $x(k) = \bar{x}$.

Case (ii): Since $\bar{x}$ is the supremum of X, for sufficiently small $\epsilon > 0$, $\bar{x} - \epsilon \in X$. That is, $\bar{\theta}(\bar{x} - \epsilon) = 1$. By Claim 4, $\forall x'' \in (0, \bar{x} - \epsilon], \bar{\theta}(x'') = 1$. Then, we set $x(k) = \bar{x} - \epsilon$.

Therefore, since $k > 0$ is arbitrary, we can conclude that there exists an $x(k)$ such that for all $x' \in (0, x(k)], \bar{\theta}(x') = 1$. ■

Proof of Proposition 6

By Proposition 5, for any $k > 0$, there exists $x(k) > 0$ such that $\forall x \in (0, x(k)), \bar{\theta}(x) = 1$. In other words, for any $k > 0$, by taking sufficiently small $x > 0$, we can find a fully separating equilibrium in the almost-common interest almost-cheap talk game. Let x_k be a preference bias supporting the fully revealing equilibrium under the magnitude of signaling costs k. Then we can construct a sequence such that $\forall k > 0, 0 < x_k < x(k)$ and $k > k'$ implies $x_k > x_{k'}$. Hence, by the construction of the sequence, $\beta^{kx}(\theta) = \beta^*(\theta) = a^R(\theta), \forall \theta \in \Theta$. and then $\beta^{kx} \to \beta^*$ pointwise as both $k \to 0$ and $x \to 0$. That is, the sequence is a desirable one. ■

Proof of Proposition 7

Consider the following sequence; for large enough n, $k_n > 0$ is sufficiently small, $k_n > k_{n+1}$, and the number of the associated equilibrium segments, N_{k_n}, satisfies $N_{k_i} < N_{k_j}$ if $i < j$. By Corollary 2, for any natural number N, there exists a partition equilibrium with N segments, so we can find the sequence. By the construction, $\lim_{k \to 0} N_k = \infty$. By Theorem 2, there exist no semi-separating and partition equilibria with countably infite segments in the common interest pure cheap talk game. Thus, the limit should be a fully separating equilibrium. ■

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