Equilibrium Selection in Persuasion Games with Binary Actions *

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Abstract

This paper studies equilibrium selection in persuasion games where the receiver's actions are binary, and discusses how to justify the most informative equilibrium as a reasonable consequence. In general, there exist multiple equilibria in this environment even if the sender's private information is fully certifiable, and the convention of focusing on the most informative equilibrium is followed without formal justification. However, we show that the existing selection criteria in the literature on strategic communication hardly justifies such a convention; in particular, these criteria might select the least informative equilibrium. We then suggest the notion of *certifiable dominance*, and show that the most informative equilibrium is uniquely selected by a perfect Bayesian equilibrium constructed using certifiably undominated strategies. This criterion could also uniquely select the most informative equilibrium when the sender's private information is partially certifiable.

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1 Introduction

As formalized by Milgrom (1981) and Milgrom and Roberts (1986), persuasion games are costless sender–receiver games, where the sender's private information is certifiable.[1] The sender has an incentive to manipulate the information transmitted to the receiver, but misrepresentation of the information is prohibited because of its certifiability. Instead of lying, the sender manipulates it by concealing unfavorable information. In other words, the sender strategically decides to what extent the information is disclosed to the receiver. These seminal papers consider an environment where (i) the sender's preference is monotonic in the receiver's action, (ii) the receiver can distinguish whether the sender conceals information, and (iii) no player can commit any strategy. They show that full information revelation is the unique equilibrium outcome, which is well-known as the *unraveling argument* in the literature.

While there exist multiple equilibria once the above assumptions are relaxed, equilibrium selection in persuasion games is not as widely investigated in the literature. Following these seminal papers, most of the existing studies check the validity of the unraveling argument under several environments.[2] That is, they conventionally focus on the fully revealing equilibrium, namely, the most informative one, among multiple equilibria. However, to the best of our best knowledge, there is no reasonable justification for such a convention. In contrast with cheap-talk games à la Crawford and Sobel (1982), the players may conflict even ex ante. Furthermore, if the fully revealing equilibrium never exists, then equilibrium selection becomes more subtle. For example, Forges and Koessler (2008) and Miura (2014) consider environments in which the sender's private information is fully certifiable but assumption (i) is relaxed, and then, they characterize the set of equilibria.[3] These papers demonstrate the nonexistence of a fully revealing equilibrium with multiple equilibria, but they do not discuss which equilibrium is the most plausible. This issue heavily restricts the applicability of persuasion games, and thus, it is important to overcome this limitation.

In this paper, we restrict our attention to an environment where the receiver's available actions

[1]Following the convention, we treat the sender as male and the receiver as female.

[2]Subsequent studies are categorized as follows. First, Seidmann and Winter (1997), Giovannoni and Seidmann (2007), and Hagenbach et al. (2014) relax assumption (i) by allowing the state-dependent preference of the sender. Second, as a relaxation of assumption (ii), Dye (1985), Jung and Kwon (1988), Okuno-Fujiwara et al. (1990), Shin (1994), Koessler (2003), and Bhattacharya and Mukherjee (2013) consider models where the sender is imperfectly informed. Likewise, Lipman and Seppi (1995), Wolinsky (2003), Lanzi and Mathis (2008), Mathis (2008), and Dziuda (2011) assume that the sender's private information is partially certifiable. Finally, Glazer and Rubinstein (2006) and the literature of Bayesian persuasion à la Kamenica and Gentzkow (2011) relax assumption (iii) by allowing commitments by either the receiver or the sender, respectively.

[3]The characterization in these papers are based on the following additional restrictions. Forges and Koessler (2008) geometrically characterize a set of equilibria requiring that the sender's certifiability is sufficiently rich. Miura (2014) focuses on a scenario where the receiver's actions are binary. To the best of our knowledge, full characterization of an equilibrium set without additional requirements is yet to be attempted.

are binary, and then challenge the issue. That is, we discuss equilibrium selection of persuasion games with binary actions, and provide justification for the convention focusing on the most informative equilibrium. Focusing on an environment with binary actions seems restrictive theoretically, but it is important to investigate this class because it includes many interesting examples, as follows.[4]

Example 1 Consider a sales clerk (sender) and a buyer (receiver) in a retail store. The buyer, who is not familiar with the specifications of tablet computers, asks the sales clerk to recommend a suitable product for her. The sales clerk suggests a product and demonstrates its performance to support his opinion. The buyer knows that the sales clerk could be biased toward some particular product, and then decides whether to "buy the recommended product" or "not."

Example 2 Consider a subordinate (sender) and a boss (receiver) in a firm. The subordinate has better information about the status of the market than the boss, and then suggests a new project by displaying data that support his suggestion. Given the preference of the subordinate toward that project, the boss decides whether to "accept" or "reject" the proposal.

Example 3 Consider a media outlet (sender) and a voter (receiver) in an election. The media outlet provides several election-related bits of information that are based on the facts, such as commentaries on proposed policies, endorsement of /opposition against the candidates, and so on. The voter discounts what the outlet conveys depending on his ideological position, and then casts the ballot for either the "Republican Party" or the "Democratic Party."

Besides these examples, the structure of persuasion games with binary actions is frequently observed in the real world. However, such a structure complicates the issue in the sense that multiple equilibria are more likely to exist even though the players' preferences are reasonable compared to a scenario without the binary structure. Thus, it is difficult to obtain a sharp prediction for such prevalent events without clarifying which equilibrium is the most plausible.

The results of this paper are as follows. First, we show that it is difficult to justify the convention that focuses on the most informative equilibrium using the selection criteria given in the literature on strategic communication. For example, the *neologism proofness* (Farrell, 1993) eliminates noting, and the *(strongly) announcement proofness* (Matthews et al., 1991) could uniquely selects the least informative equilibrium if the fully pooling equilibrium exists; otherwise, all equilibria are

[4] Events similar to the examples can be modeled as persuasion games. See, for example, Celik (2014), Itoh (2016), and Miura (2015), respectively.

eliminated. Second, we suggest the notion of *certifiable dominance* as a selection criterion, and then show that the most informative equilibrium is uniquely selected by a perfect Bayesian equilibrium (hereafter, PBE) constructed by certifiably undominated strategies. The certifiable dominance is the modified version of weak dominance in the sense that the players' strategies are restricted to those consistent with the certification assumption when we apply the dominance argument. Hence, this notion is closely related to Δ-*rationalizability* à la Battigalli and Siniscalchi (2003) in the sense that the Δ-rationalizability derives the same selection results under the appropriate restrictions on beliefs. Finally, although we mainly focus on the scenario where the sender's private information is fully certifiable, we show that the unique selection of the most informative equilibrium by the certifiable dominance can be extended to the scenario of partial certifiable information.

This paper is organized as follows. In the next subsection, we discuss the related literature. In Section 2, we outline the model where the sender's private information is fully certifiable. In Section 3, we review the characterization of the equilibrium set, and then discuss equilibrium selection in Section 4. In Section 5, we extend the selection result to an environment with partially certifiable information. Section 6 concludes the paper. All the proofs appear in Appendix A.

1.1 Related literature

It is common practice to restrict players' choices to a binary decision is widespread in many fields of economics, including the literature on persuasion games. In the context of persuasion games, restrictions on the binary decision by the receiver enable us to provide a full characterization of equilibria, optimal decision rules, and so on. First, Lanzi and Mathis (2008), Dziuda (2011), and Miura (2014) describe non-fully-revealing behaviors in equilibrium, assuming that the players do not make commitments. Lanzi and Mathis (2008) and Dziuda (2011), on the one hand, consider persuasion games in which the sender's private information is partially certifiable, and provide detailed properties of the equilibria. On the other hand, Miura (2014) assumes fully certifiable information as the cost of fewer restrictions on preferences, and characterizes the set of equilibria in terms of the receiver's ex ante expected utility. Second, Glazer and Rubinstein (2006), Rayo and Segal (2010), and Kolotilin (2015) analyze models where the players can commit their behaviors. Glazer and Rubinstein (2006) allow commitments to decision rules by the receiver, and characterize the optimal decision rule that minimizes the probability of incorrect decision making by the receiver. In the context of Bayesian persuasion, Rayo and Segal (2010) and Kolotilin (2015) provide a detailed characterization of the optimal disclosure rule that maximizes the sender's utility.

Equilibrium selection is one of the main concerns in the literature on signaling games. In costly signaling games, Cho and Kreps (1987) and Banks and Sobel (1987) propose well-known criteria, that is, the *intuitive criterion*, *D1/D2 criterion*, and *(universal) divinity*, based on the concept of *strategic stability* proposed by Kohlberg and Mertens (1986), which typically selects the separating equilibria. In contrast, *undefeatedness*, developed by Mailath et al. (1993), is immune to the Stiglitz critique, and could select a pooling equilibrium as a reasonable prediction. In cheap-talk games, Farrell (1993) develops the notion of neologism proofness, which is justified by self-fulfilling beliefs as in Grossman and Perry (1986), and Matthews et al. (1991) propose the announcement proofness as a generalization of the neologism proofness. On the other hand, *credible message rationalizability* by Rabin (1990) and *proposal proofness* by Zapater (1997) are criteria based on the rationalizability. While these criteria work well in some classes of cheap-talk games, they are useless under Crawford and Sobel's (1982) framework. As a useful criterion for the Crawford-Sobel games, Chen et al. (2008) propose the *NITS condition* that uniquely selects the most informative equilibrium under some regularity conditions.

This paper contributes to the literature as follows. We borrow the models of Miura (2014) and Lanzi and Mathis (2008), and discuss which equilibrium is the most plausible or how to justify the convention focusing on the most informative equilibrium. In contrast with costly signaling and cheap-talk games, equilibrium selection in persuasion games is less discussed in the literature. As a few exceptions, Giovannoni and Seidmann (2007) and Chen et al. (2008) informally discuss the limitation of the intuitive criterion and the NITS condition, respectively. Ryan and Vaithianathan (2011) construct an example to show that the fully revealing equilibrium is eliminated by the neologism proofness. This paper provides formal and comprehensive arguments on equilibrium selection in persuasion games with binary actions.

2 The Model

There exists one sender and one receiver. The receiver has to choose action $y \in Y \equiv \{y_1, y_2\}$, but the outcome depends on the state of nature $\theta \in \Theta \equiv [0,1]$, which is the sender's private information. Let $F(\cdot)$ be the atomless common prior distribution over state space Θ with full support and density function $f(\cdot)$. Let $m \in M(\theta) \equiv \left\{ m \in 2^\Theta \;\middle|\; \theta \in m \right\}$ be a message from the sender, where $M(\theta)$ is the sender's message space when the state is θ. Denote the entire message space by $M \equiv \bigcup_{\theta \in \Theta} M(\theta)$, and the set of states where message m is available by $M^{-1}(m) \equiv \{ \theta \in \Theta \mid m \in M(\theta) \}$. There are two remarks to be mentioned. First, any available message under state θ must contain the truth. Second, for any subset $T \subseteq \Theta$, message $m = T$ has the property that $M^{-1}(T) = T$; that is, any state is *fully certifiable.*

The players' preferences are defined as follows. The receiver's and the sender's von Neumann–Morgenstern utility functions are denoted by $u : \Theta \times Y \to \mathbb{R}$ and $v : \Theta \times Y \to \mathbb{R}$, respectively. The state space Θ is divided into the following regions depending on ex post conflicts between the players. Let $\Theta_{ij} \equiv \left\{ \theta \in \Theta \,\middle|\, u(\theta, y_i) > u(\theta, y_{i'}) \text{ and } v(\theta, y_j) > v(\theta, y_{j'}) \right\}$ be the set of states in which the receiver strictly prefers action y_i to $y_{i'}$ and the sender strictly prefers action y_j to $y_{j'}$ ex post for $i, i', j, j' \in \{1, 2\}$ with $i \neq i'$ and $j \neq j'$, and define $\Theta_0 \equiv \Theta \setminus (\Theta_{11} \cup \Theta_{22} \cup \Theta_{12} \cup \Theta_{21})$. If state θ lies in regions $\Theta_{11} \cup \Theta_{22} \cup \Theta_0$, then the players' ex post preferred actions coincide. Otherwise, their ex post preferred actions are in conflict. We call the former *agreement regions* and the latter *disagreement regions*, respectively. Let $y^R(\theta) \in \arg\max_{y \in Y} u(\theta, y)$ be the receiver's ex post preferred action at state θ. Likewise, let $y^S(\theta)$ represent the sender's ex post preferred action at θ. Hereafter, to simplify representations, for measurable $T \subseteq \Theta$, we denote $P(T) \equiv \int_T f(\theta) d\theta$, and $\mathbb{E}[\cdot|T] \equiv \mathbb{E}[\cdot|\theta \in T]$. We assume that all information except state θ is common knowledge.

We assume the following for the preferences.

Assumption 1

(i) $P(\Theta_0) = 0$.

(ii) $\mathbb{E}\left[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}\right] \neq \mathbb{E}\left[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}\right]$.

Assumption 1 is a mild requirement. Condition (i) means that either one of the players being indifferent between actions is a measure-0 event, and Condition (ii) is generically satisfied. To ease exposition, we, hereafter, assume that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] > \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$, i.e., the receiver strictly prefers action y_1 conditional on the entire disagreement region, without loss of generality. These conditions are essential for both the characterization of the equilibrium set and equilibrium selection.

Assumption 2

(i) Both u and v are continuous in θ for any $y \in Y$.

(ii) $\mathbb{E}\left[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}\right] \geq \mathbb{E}\left[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}\right]$ *and* $\mathbb{E}\left[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}\right] \leq \mathbb{E}\left[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}\right]$.

It is worthwhile to mention that Assumption 2 only simplifies the characterization of the equilibrium set, and it is irrelevant to the equilibrium selection.[5] Condition (ii) requires that ex post conflict is

[5] As long as we focus on pure strategy equilibria, we do not require Condition (ii). Combined with Assumptions 1 and 2-(ii), the most and the least informative equilibria are characterized by pure strategies even though mixed strategies are allowed. Hence, we can focus on pure strategy equilibria without loss of generality when we characterize the equilibrium set. See Corollary 2 of Miura (2014).

not too strong in the sense that the receiver also weakly prefers action y_1 (resp. y_2) conditional on the overall states in which the sender strictly prefers action y_1 (resp. y_2).

The timing of the game is as follows. First, nature chooses the state of the world $\theta \in \Theta$ according to the prior distribution $f(\cdot)$, and only the sender observes it. Given state θ, the sender sends a message $m \in M(\theta)$. After observing message m, the receiver undertakes an action $y \in Y$.

The players' strategies and beliefs are defined as follows. Let $\sigma : \Theta \to \Delta^*(M)$ be the sender's strategy where $\Delta^*(M)$ represents the set of finite-support probability distributions over M.[6] Let $\phi : M \to \Delta(Y)$ and $\mathcal{P} : M \to \Delta(\Theta)$ be the receiver's strategy and posterior belief, respectively.[7] We use the PBE as a solution concept. Because of the full certifiability, we put the following additional restrictions on off-the-equilibrium-path beliefs, which are common in the literature. We denote the support of probability distribution g by $S(g)$.

Requirement 1 *For any message $m \in M$, $S(\mathcal{P}(m)) \subseteq m$.*

Definition 1 ***PBE***
A triple $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a PBE if it satisfies the following conditions:

(i) For any $\theta \in \Theta$ and $m \in S(\sigma^*(\theta))$, $m \in \arg\max_{m' \in M(\theta)} v\left(\theta, \phi^*(m')\right)$;

(ii) For any $m \in M$ and $y \in S(\phi^*(m))$, $y \in \arg\max_{y' \in Y} \mathbb{E}_{\mathcal{P}^*(m)}\left[u(\theta, y')\right]$;

(iii) $\mathcal{P}^*$ is derived by σ^* consistently from Bayes' rule whenever it is possible. Otherwise, $\mathcal{P}^*$ is some probability distribution satisfying Requirement 1.

We evaluate each equilibrium in terms of its informativeness, measured by the receiver's ex ante expected utility. Let $\bar{U}(\theta, \phi(\sigma(\theta)))$ represent the receiver's expected utility at state θ if strategy (σ, ϕ) is played, i.e.,

$$\bar{U}(\theta, \phi(\sigma(\theta))) \equiv \sum_{m \in M} \sum_{y \in Y} u(\theta, y)\phi(y|m)\sigma(m|\theta), \tag{1}$$

and $\mathbb{E}[\bar{U}(\theta, \phi^*(\sigma^*(\theta)))]$ represent the receiver's ex ante expected utility in equilibrium $(\sigma^*, \phi^*; \mathcal{P}^*)$. We say that PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a *fully revealing equilibrium* if $\phi^*(\sigma^*(\theta)) = y^R(\theta)$ for any $\theta \in \Theta$.

[6]For technical convenience, we exclude mixed strategies whose supports are infinite sets.
[7]With some abuse of notation, pure strategies of the sender and receiver are simply represented by $\sigma(\theta) = m$ and $\phi(m) = y$, respectively.

3 Set of Equilibria

The set of equilibria is characterized by the most/least informative equilibria. Then there exist continuum equilibria as follows.[8]

Theorem 1 *(Miura, 2014)* *Suppose that Assumptions 1 and 2 hold.*

(i) One of the most informative equilibria $(\sigma^+, \phi^+; \mathcal{P}^+)$ *with* $\mathbb{E}\left[\bar{U}(\theta, \phi^+(\sigma^+(\theta)))\right] = U^+$ *is as follows:*

$$\sigma^+(\theta) = \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0, \\ \Theta_{12} \cup \Theta_{21} & \text{otherwise.} \end{cases} \tag{2}$$

(ii) One of the least informative equilibria $(\sigma^-, \phi^-; \mathcal{P}^-)$ *with* $\mathbb{E}\left[\bar{U}(\theta, \phi^-(\sigma^-(\theta)))\right] = U^-$ *is as follows:*

$$\sigma^-(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21}, \\ \Theta_{22} \cup \Theta_{12} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12}, \\ \{\theta\} & \text{otherwise.} \end{cases} \tag{3}$$

(iii) There exists equilibrium $(\sigma, \phi; \mathcal{P})$ *such that* $\mathbb{E}[u(\theta, \phi(\sigma(\theta)))] = U$ *if and only if* $U \in [U^-, U^+]$.

Proof. See Miura (2014). ■

In the most informative equilibrium $(\sigma^+, \phi^+; \mathcal{P}^+)$, all types in the agreement regions fully disclose their own identities, and the others send a pooling message. This structure is associated with the fully revealing equilibrium if it exists.[9] On the other hand, in the least informative equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$, only two messages are generically sent on the equilibrium path. That is, almost every type who strictly prefers action y_1 (resp. y_2) sends message $\Theta_{11} \cup \Theta_{21}$ (resp. $\Theta_{22} \cup \Theta_{12}$).[10] Notice that any type in the agreement regions $\Theta_{11} \cup \Theta_{22}$ should certainly induce his ideal action that is identical to $y^R(\theta)$ in any equilibrium; otherwise, such a type has an incentive to fully disclose his type. Hence, the informativeness of each equilibrium is essentially determined

[8] It is worth mentioning that while this measurement is widely used in the literature on strategic communication games, e.g., Crawford and Sobel (1982), it is different from a similar-sounding concept used in information theory.

[9] The fully revealing equilibrium exists if and only if either $\Theta_{12} = \emptyset$ or $\Theta_{21} = \emptyset$ holds. If $\Theta_{12} \neq \emptyset$ and $\Theta_{21} \neq \emptyset$, then type $\theta \in \Theta_{12}$ has an incentive to mimic type $\theta' \in \Theta_{21}$ and vice versa. Hence, the fully revealing equilibrium never exists. See Giovannoni and Seidmann (2007), Hagenbach et al. (2014), and Miura(2014).

[10] Without Assumption 2-(ii), such a pooling structure is never supported in equilibrium. However, we can construct an equilibrium with a similar structure, which minimizes the receiver's ex ante expected utility. See Proposition 3 of Miura (2014).

depending on the equilibrium outcomes over the disagreement regions. It is then straightforward that equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$ minimizes the receiver's ex ante expected utility because the receiver undertakes ex post incorrect action over the entire disagreement region. It is worthwhile to remark that the players' preferences conflict even ex ante in the sense that the least informative equilibrium is the best scenario for the sender. Because of the full certifiability, we can construct an equilibrium in which the receiver undertakes ex post incorrect action over any region X such that $\Theta_{21} \subseteq X \subseteq \Theta_{12} \cup \Theta_{21}$, which is the main reason for the existence of continuum equilibria.

4 Equilibrium Selection

This section studies the equilibrium selection in this environment. First, we check the validity of the existing criteria. We then suggest the notion and usefulness of certifiable dominance.

4.1 Existing criteria

In this subsection, we discuss the validity of the neologism proofness and announcement proofness, both of which are well-known criteria in the literature on cheap-talk games.[11] These criteria seem applicable even in the context of persuasion games. When applying these criteria in cheap-talk games, we additionally assume that some messages have literal meaning; that is, the meaning of these messages is exogenously given, which differs from cheap-talk messages whose meaning is endogenously determined in equilibrium. Because the sender's message space varies depending on his type in persuasion games, there exists a message that is available to type θ but unavailable to type θ'. The literal meaning of such a message is that "the sender's type is not θ'." Hence, the assumption of the literal meaning generally holds in persuasion games, and then we can apply these criteria without additional assumptions. We show that the neologism proofness has no bite in this environment, but the announcement proofness uniquely selects the least informative equilibrium under Assumption 2, which is incoherent with the convention focusing on the fully revealing equilibrium.[12]

We introduce additional notation. Let $d \equiv (\delta, D)$ be an *announcement strategy* where (i) a

[11] The NITS condition (Chen et al., 2008) is also a well-known criterion in the literature on cheap-talk games. However, this criterion is specialized to the class of cheap-talk games developed by Crawford and Sobel (1982), and the crucial assumptions in that class may not be satisfied in our environment. Hence, it seems inappropriate to apply the NITS condition to our setup.

[12] In Appendix B, we also discuss the validity of (i) the intuitive criterion (Cho and Kreps, 1987), (ii) D1 and D2 criteria (Cho and Kreps, 1987; Cho and Sobel, 1990), (iii) undefeatedness (Mailath et al., 1993), (iv) credible message equilibrium (Rabin, 1990), and (v) proposal-proof equilibrium (Zapater, 1997). We show that (i), (ii), (iii), and (v) eliminate noting like the neologism proofness. On the other hand, (iv) uniquely selects the least informative equilibrium under a slight modification.

nonempty subset $D \subseteq \Theta$ is a set of *deviant types*, and (ii) a function $\delta : D \to \Delta^*(M)$ with $\delta(\theta) \in \Delta(M(\theta))$ for any $\theta \in D$ is a *talking strategy* of the deviant types.[13] The set of announcement strategies is denoted by Σ_d. We say that an announcement strategy is a *neologism* if $\delta(\theta) = D$ for any $\theta \in D$. Let $a \equiv (m, d)$ such that $m \in S\left(\bigcup_{\theta \in D} \delta(\theta)\right)$ denote an *announcement.* Let $\mathcal{P}_A : \Theta \times M \times \Sigma_d \to \Delta(\Theta)$ be the receiver's consistent belief after observing announcement (m, d), which is defined by:

$$\mathcal{P}_A(\theta|m,d) \equiv \begin{cases} \dfrac{f(\theta)\delta(m|\theta)}{\int_D f(\hat{\theta})\delta(m|\hat{\theta})d\hat{\theta}} & \text{if } \theta \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let $BR_R(m, \mathcal{P})$ be the set of the receiver's best responses, given message m and posterior belief $\mathcal{P}$, i.e.,[14]

$$\begin{aligned} BR_R(m,\mathcal{P}) &\equiv \arg \max_{\phi(\cdot|m)\in\Delta(Y)} \int_\Theta \sum_{y\in Y} \phi(y|m)u(\theta,y)\mathcal{P}(\theta|m)d\theta \qquad (5) \\ &= \arg \max_{\alpha\in[0,1]} \int_\Theta \left(\alpha u(\theta,y_1) + (1-\alpha)u(\theta,y_2)\right)\mathcal{P}(\theta|m)d\theta. \end{aligned}$$

Let $V(\theta, \alpha)$ denote the expected utility of type θ when the receiver chooses action y_1 with probability $\alpha \in [0, 1]$ defined by:

$$V(\theta,\alpha) \equiv \alpha v(\theta,y_1) + (1-\alpha)v(\theta,y_2). \tag{6}$$

Likewise, let $\bar{V}(\theta, \phi(\sigma(\theta)))$ denote type θ's expected utility, given the strategy pair (σ, ϕ) defined by:

$$\bar{V}(\theta,\phi(\sigma(\theta))) \equiv \sum_{m\in M}\sum_{y\in Y} v(\theta,y)\phi(y|m)\sigma(m|\theta). \tag{7}$$

The neologism-proof equilibrium is defined as follows.

Definition 2 *Neologism-Proof Equilibrium (Farrell, 1993)*

(i) Neologism (d, D) is credible relative to PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ if the following conditions hold:

(N0) $D \notin S(\sigma^*(\theta))$ for any $\theta \in \Theta$;[15]

[13] To economize notation, a degenerate distribution is simply represented by $\delta(\theta) = m$ with some abuse of notation.

[14] If the set of the best responses is a singleton set, then we simply represent it by $BR_R(m, \mathcal{P}) = \{y_i\}$ with some abuse of notation.

[15] This additional condition is needed for our environment because there may not exist appropriate off-the-

(N1) $V(\theta, \alpha) > \bar{V}(\theta, \phi^*(\sigma^*(\theta)))$ *for any* $\theta \in D$ *and* $\alpha \in BR_R(D, \mathcal{P}_A)$*; and*

(N2) $V(\theta, \alpha) \leq \bar{V}(\theta, \phi^*(\sigma^*(\theta)))$ *for any* $\theta \in \Theta \backslash D$ *such that* $D \in M(\theta)$ *and* $\alpha \in BR_R(D, \mathcal{P}_A)$*.*

(ii) A PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ *is neologism proof if no neologism is credible relative to it.*

Intuitively, the credible neologism requires that given a neologism claiming that "my type is in set D" and the receiver believes that neologism, (i) all types included in deviant type set D have an incentive to send such a neologism, but (ii) the other types have no incentive.[16] The neologism proofness insists that a PBE that is immune to such credible neologisms is a reasonable consequence, but it does not work at all in our environment as shown in the following proposition.

Proposition 1 *Any PBE is neologism proof.*

The ineffectiveness of the neologism proofness comes from the fact that all deviant types must obtain strictly higher utility from sending a neologism. However, it is a demanding requirement in our setup. The set of the potentially deviant types must be a subset of Θ_{12} and Θ_{21} in any equilibrium, and such types cannot obtain strictly higher utility if the receiver rationally responds to neologisms. In other words, there is no credible neologism in our environment. This problem can be avoided by considering announcement proofness, which is the generalization of neologism proofness.

Definition 3 *Strongly Announcement-Proof Equilibrium (Matthews et al., 1991)*

(i) An announcement (m, d) *is weakly credible relative to PBE* $(\sigma^*, \phi^*; \mathcal{P}^*)$ *if the following conditions hold:*

(A0) $m \notin S(\sigma^*(\theta))$ *for any* $\theta \in \Theta$*;*

(A1) $\min_{\alpha \in BR_R(m, \mathcal{P}_A)} V(\theta, \alpha) \geq \bar{V}(\theta, \phi^*(\sigma^*(\theta)))$ *for any* $\theta \in D$ *and* $m \in S(\delta(\theta))$ *with the strict inequality for some* $\hat{\theta} \in D$ *and* $\hat{m} \in S(\delta(\hat{\theta}))$*;*

(A2) $\max_{\alpha \in BR_R(m, \mathcal{P}_A)} V(\theta, \alpha) \leq \bar{V}(\theta, \phi^*(\sigma^*(\theta)))$ *for any* $\theta \in \Theta \backslash D$ *and* $m \in M(\theta) \cap \left(\bigcup_{\theta' \in D} S(\delta(\theta')) \right)$*;*

(A3) $\min_{\alpha \in BR_R(m, \mathcal{P}_A)} V(\theta, \alpha) \geq \max_{\alpha' \in BR_R(m', \mathcal{P}_A)} V(\theta, \alpha')$ *for any* $\theta \in D$ *and* $m \in S(\delta(\theta))$ *and* $m' \in \left(\bigcup_{\theta' \in D} S(\delta(\theta')) \backslash \{m\} \right) \cap M(\theta)$*.*

equilibrium-path messages that can be used as neologisms.

[16]Because of the full certifiability assumption, for any type $\theta \in \Theta \backslash D$, $D \notin M(\theta)$. Hence, Condition (ii) is vacuously satisfied.

(ii) A PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ is strongly announcement proof if no announcement is weakly credible relative to it.

The (strongly) announcement proofness extends the neologism proofness in the sense that the deviant types could send multiple off-the-equilibrium-path messages.[17] Furthermore, it is not necessary that all deviant types obtain a strictly higher utility than the equilibrium utility. Because of this generalization, the strongly announcement proofness could work well in our environment, as shown in the following proposition.

Proposition 2

(i) Suppose that Assumptions 1 and 2 hold. Then, U^- is the unique informativeness selected by the strongly announcement proofness.

(ii) Suppose that Assumption 2-(ii) does not hold. Then, there exists no strongly announcement-proof equilibrium.[18]

Certain observations are notable. First, the strongly announcement proofness selects the least informative equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$ because this equilibrium is the best scenario for the sender. As any type of sender induces his ideal action in equilibrium, no announcement is weakly credible. Second, while the strongly announcement proofness could work well in our environment, this selection result is incoherent with the convention of focusing on the fully revealing equilibrium. That is, if the fully revealing equilibrium exists, then it is trivially the most informative equilibrium. However, focusing on the most informative equilibrium is not supported by the strongly announcement proofness under Assumption 2-(ii). Finally, this selection result does not generally hold. Without Assumption 2-(ii), equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$ does not exist; that is, there exists type $\theta \in \Theta_{12} \cup \Theta_{21}$ such that $\phi^*(\sigma^*(\theta)) = y^R(\theta)$ in any equilibrium. Because such a type can send a credible announcement, no equilibrium survives.[19]

4.2 Certifiable dominance

The less informative equilibria are justified by the existing criteria, but they seems unreasonable in the sense that they are constructed by a sense of dominated strategies. For example, in the

[17] Notice that the credible neologism is a weakly credible announcement but the converse may not be true. Hence, the strongly announcement proofness is a refinement of the neologism proofness.

[18] Matthews et al. (1991) also define the *announcement proofness* and the *weakly announcement proofness*, which are weaker than the strongly announcement proofness. Even if we adopt those weaker versions, the selection results do not change. The corresponding details are available from the author upon request

[19] Hedlund (2015) applies the strongly announcement proofness to a model where the sender has to pay a higher cost if he sends more precise messages. In contrast with our case, the strongly announcement proofness could select both the fully separating and the fully pooling equilibria.

least informative equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$ characterized in Theorem 1, type $\theta \in \Theta_{11}$ pools with other types in Θ_{21}. However, such a pooling message is weakly dominated by the fully revealing message, i.e., $m = \{\theta\}$, in the following sense. Notice that, on the one hand, the fully revealing message completely identifies the sender's type, and then the receiver's best response is uniquely determined, namely, $y = y_1 = y^S(\theta)$. On the other hand, the receiver could react to the pooling message by choosing action $y = y_2$ if she believes that the pooling message is more likely to be sent by the types in Θ_{21}. This subsection proposes a selection criterion that eliminates equilibria in which such a dominated message is used, and shows that it uniquely selects the most informative equilibrium.

We formalize the criterion as follows. Let Σ and Φ be the sets of the sender's and receiver's strategies defined by $\Sigma \equiv \left\{ \sigma \in \Delta^*(M)^\Theta \;\middle|\; m \in M(\theta) \; \forall\theta \in \Theta \text{ and } m \in S(\sigma(\theta)) \right\}$ and $\Phi \equiv \Delta(Y)^M$, respectively. For $\Sigma' \subseteq \Sigma$ and $\Phi' \subseteq \Phi$, we say that strategy $\sigma \in \Sigma'$ is *dominated in* $\Sigma' \times \Phi'$ if there exists strategy $\sigma' \in \Sigma'$ such that $\bar{V}(\theta, \phi(\sigma'(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ for any $\theta \in \Theta$ and $\phi \in \Phi'$ with strict inequality for some $\theta' \in \Theta$ and $\phi' \in \Phi$. We define:

$$\Phi_C \equiv \left\{ \phi \in \Phi \;\middle|\; \forall m \in M,\, \phi(m) \in \Delta\left(\bigcup_{\theta \in M^{-1}(m)} \{y^R(\theta)\} \right) \right\}, \tag{8}$$

and restrict our attention to Φ_C when we apply the weak dominance. We then select PBEs that are supported by "undominated" strategies, in the above sense, of the sender as reasonable ones.

Definition 4 *Certifiably Undominated Equilibrium (hereafter, CUE)*

(i) The sender's strategy σ is certifiably undominated if σ is undominated in $\Sigma \times \Phi_C$. Let Σ_C be the set of the certifiably undominated strategies of the sender.

(ii) A PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a CUE if $\sigma^* \in \Sigma_C$.

Intuitively, the certifiable dominance is a version of weak dominance that is consistent with the rationality of the players and the assumption of certifiable states. Because any state is fully certifiable, the receiver certainly learns that some states never occur after observing a message. Hence, the rational receiver should never choose actions that are preferred ex post under states inconsistent with the observed message. For instance, if the receiver observes message $m \subseteq \Theta_{11}$, then she learns that the true state is in Θ_{11}, and so, choosing action $y = y_2$ with positive probability cannot be justified as the best response to message m. That is, Φ_C is the set of the receiver's strategies that are consistent with her rationality and the certification assumption. Because the sender knows that the receiver is rational and she understands the literal meaning of the messages, it seems reasonable to restrict the attention to Φ_C when we apply the dominance criterion on the sender's strategies. In addition, the weak dominance has no bite in our environment without the restriction to Φ_C; that is, any strategy $\sigma \in \Sigma$ is undominated in $\Sigma \times \Phi$. Thus, this restriction can be regarded as a modification of the weak dominance so that it may have a bite in this environment.

The set of certifiably undominated strategies Σ_C is characterized as follows. Define $\Theta_{10} \equiv \{\, \theta \in \Theta \mid u(\theta, y_1) > u(\theta, y_2) \text{ and } v(\theta, y_1) = v(\theta, y_2) \,\}$, and Θ_{20}, Θ_{01}, Θ_{02}, and Θ_{00} are analogously defined.

Proposition 3

$$\Sigma_C = \hat{\Sigma} \equiv \left\{ \sigma \in \Sigma \;\middle|\; \begin{array}{l} \forall m \in S(\sigma(\theta)), \\ m \begin{cases} \subseteq \Theta_{11} \cup \Theta_{12} \cup \Theta_{10} & \textit{if } \theta \in \Theta_{11}, \\ \subseteq \Theta_{22} \cup \Theta_{21} \cup \Theta_{20} & \textit{if } \theta \in \Theta_{22}, \\ \cap (\Theta_{22} \cup \Theta_{21} \cup \Theta_0 \backslash \Theta_{10}) \neq \emptyset & \textit{if } \theta \in \Theta_{12}, \\ \cap (\Theta_{11} \cup \Theta_{12} \cup \Theta_0 \backslash \Theta_{20}) \neq \emptyset & \textit{if } \theta \in \Theta_{21} \end{cases} \end{array} \right\}. \tag{9}$$

Intuitively, the certifiably undominated strategies require that (i) types in Θ_{11} (resp. Θ_{22}) never sends messages including states in which the receiver weakly prefers action y_2 (resp. y_1), and (ii) types in Θ_{12} must send messages including states in which the receiver weakly prefers action y_2 (resp. y_1). We must note the following two points. First, the most informative equilibrium $(\sigma^+, \phi^+; \mathcal{P}^+)$ is a CUE. From Proposition 3, it is obvious that σ^+ is certifiably undominated. Hence, the set of CUEs is nonempty. Second, on the other hand, the least informative equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$ is not a CUE. It is also obvious that σ^- is not certifiably undominated because $\sigma^-(\theta) \nsubseteq \Theta_{11} \cup \Theta_{12} \cup \Theta_{10}$ for any $\theta \in \Theta_{11}$. Therefore, as long as we focus on CUEs, we can successfully eliminate the least informative equilibrium.

It is worthwhile to emphasize that the power of certifiable dominance lies in not only eliminating the least informative equilibrium, but also eliminating all PBEs except for the most informative equilibrium. That is, the CUE uniquely selects the most informative equilibrium as a reasonable prediction of the model. The key insight deriving the uniqueness is that almost every type in the disagreement regions $\Theta_{12} \cup \Theta_{21}$ should induce the same action in CUEs, as shown in the following proposition.

Proposition 4 *Suppose that Assumption 1 holds. If* $(\sigma^*, \phi^*; \mathcal{P}^*)$ *is a CUE, then* $\mathbb{E}[\bar{U}(\theta, \phi^*(\sigma^*(\theta)))|\Theta_{12} \cup \Theta_{21}] = \max_{y \in Y} \mathbb{E}[u(\theta, y)|\Theta_{12} \cup \Theta_{21}]$.

Because types in the agreement regions Θ_{11} (resp. Θ_{22}) are separated from types in the disagreement regions Θ_{21} (resp. Θ_{12}) as long as the sender adopts a certifiably undominated strategy, this property is straightforward if either $P(\Theta_{12}) = 0$ or $P(\Theta_{21}) = 0$. However, this property still holds even if $P(\Theta_{12}) > 0$ and $P(\Theta_{21}) > 0$. In this scenario, most types in Θ_{12} (resp. Θ_{21}) should be pooling with types in Θ_{21} (resp. Θ_{21}). Furthermore, this property does not depend on Assumption 2-(ii), which is a key condition for the selection by the strongly announcement proofness. Combining this property with the fact that any type in the agreement regions $\Theta_{11} \cup \Theta_{22}$ induces his ideal action in any PBE implies the uniqueness in terms of the informativeness.

Theorem 2 *Suppose that Assumption 1 holds. Then,* U^+ *is the unique informativeness selected by CUEs.*

Theorem 2 causes us to make the following remarks. First, while the selection result is only dependent on Assumption 1, it implies that the most informative equilibrium is uniquely selected with Assumption 2. As we mentioned above, if we either (i) focus on pure strategy equilibria or (ii) allow mixed strategies with Assumption 2, then U^+ is the maximized informativeness. Hence, we can insist that the CUE uniquely selects the most informative equilibrium. Second, even if mixed strategies are allowed without Assumption 2, the CUE is still uniquely determined up to the informativeness. Because it may not be the maximized informativeness in this scenario, the unique selection of the most informative equilibrium may not be guaranteed. However, PBE $(\sigma^+, \phi^+; \mathcal{P}^+)$ is associated with the fully revealing equilibrium if it exists, so we can insist that the fully revealing equilibrium is the unique CUE up to the informativeness, which is coherent with the convention in the literature focusing on the fully revealing equilibrium.

4.3 Discussion: connection to rationalizability

The notion of certifiable dominance is closely related to Δ-rationalizability developed by Battigalli and Siniscalchi (2003). This is an extension of the extensive-form rationalizability by Pearce (1984) to incomplete information games, in which the players' first-order beliefs are explicitly restricted instead of specifying a epistemic type space à la Harsanyi (1967-68). We insist that the same selection result can be obtained by appropriately restricting beliefs. To ease exposition, we assume that $\Theta_{12} \neq \emptyset$ and $\Theta_{21} \neq \emptyset$ throughout this subsection.[20]

[20]The other cases are discussed in Appendix B.

We add the following notation, and, hereafter, focus on pure strategy equilibria following Battigalli (2006). Let $\bar{\Sigma}$ and $\bar{\Phi}$ be the sets of the sender's and the receiver's pure strategies defined by $\bar{\Sigma} \equiv \left\{ \sigma \in M^{\Theta} \;\middle|\; \forall \theta \in \Theta,\ \sigma(\theta) \in M(\theta) \right\}$ and $\bar{\Phi} \equiv Y^M$, respectively. Let $\mu^S \in \Delta(\bar{\Phi})$ be the sender's belief to the receiver's strategy. The receiver's belief is represented by a system of conditional probabilities $\mu^R \equiv \left(\mu^R(\phi), (\mu^R(m))_{m \in M}\right) \in \bar{\Delta}(\Theta, M) \equiv \Delta(\Theta \times M) \times \Delta(\Theta)^M$, where $\mu^R(\phi)$ denotes the receiver's initial belief and $\mu^R(m)$ is a posterior belief over the state space upon message m that satisfies the following conditions: (i) $\mu^R(\Theta(m)|m) = 1$ for any $m \in M$ where $\Theta(m) \equiv \left\{ \theta \in \Theta \;\middle|\; \mu^R(\theta, m|\phi) > 0 \right\}$, and (ii) for any $\theta \in \Theta$ and $m \in M$, if $\mu^R(\Theta \times \{m\}|\phi) > 0$, then $\mu^R(\theta|m) = \mu^R(\theta, m|\phi)/\mu^R(\Theta \times \{m\}|\phi)$. Let $BR_S : \Theta \times \Delta(\bar{\Phi}) \rightrightarrows M$ be the sender's best response correspondence defined as follows: for any $\theta \in \Theta$ and $\mu^S \in \Delta(\bar{\Phi})$,

$$BR_S(\theta, \mu^S) \equiv \arg \max_{m \in M(\theta)} \sum_{y \in Y} v(\theta, y)\pi(y|m, \mu^S), \tag{10}$$

where $\pi(y|m, \mu^S) \equiv \mu^S \left(\left\{ \phi \in \bar{\Phi} \;\middle|\; \phi(m) = y \right\}\right)$. With some abuse of notation, denote $BR_R(m, \mu^R(m))$ as the receiver's best response correspondence given message m and belief μ^R, which is defined analogous to (5) with a restriction to pure strategies. Let $\Delta \equiv (\Delta^S, \Delta^R)$ represent the explicit restrictions on beliefs where $\Delta^S \subseteq \Delta(\bar{\Phi})$ and $\Delta^R \subseteq \bar{\Delta}(\Theta, M)$. Define $\bar{\Phi}_i(m) \equiv \left\{ \phi \in \bar{\Phi} \;\middle|\; \phi(m) = y_i \right\}$ for $i = 1, 2$, and let $G(\sigma) \equiv \{ (\theta, m) \in \Theta \times M \mid \sigma(\theta) = m \}$ represent the graph of strategy σ.

The procedure of iterative elimination is defined as follows. Given restriction Δ, let $\Sigma^0_\Delta \equiv \bigcup_{\sigma \in \bar{\Sigma}} G(\sigma)$ and $\Phi^{-1}_\Delta \equiv \bar{\Phi}$ be the graph of the sender's pure strategies and the set of the receiver's pure strategy as initial points, respectively. For $n \geq 1$, Θ^{2n-1}_Δ, Φ^{2n-1}_Δ, and Σ^{2n}_Δ are recursively defined as follows:[21]

$$\Theta^{2n-1}_\Delta(m) \equiv \left\{ \theta \in \Theta \;\middle|\; (\theta, m) \in \Sigma^{2n-2}_\Delta \right\}, \tag{11}$$

$$\Phi^{2n-1}_\Delta \equiv \left\{ \phi \in \Phi^{2n-3}_\Delta \;\middle|\; \begin{array}{l} \exists \mu^R \in \Delta^R \text{ such that:} \\ \text{(i) } \forall m \in M,\ \phi(m) \in BR_R(m, \mu^R(m));\ \text{and} \\ \text{(ii) if } \Theta^{2n-1}_\Delta(m) \neq \emptyset,\ \text{then } \mu^R\left(\Theta^{2n-1}_\Delta(m)|m\right) = 1 \end{array} \right\}, \tag{12}$$

$$\Sigma^{2n}_\Delta \equiv \left\{ (\theta, m) \in \Sigma^{2n-2}_\Delta \;\middle|\; \begin{array}{l} \exists \mu^S \in \Delta^S \text{ such that:} \\ \text{(i) } m \in BR_S(\theta, \mu^S);\ \text{and (ii) } \mu^S\left(\Phi^{2n-1}_\Delta\right) = 1 \end{array} \right\}. \tag{13}$$

Finally, define $\Sigma^*_\Delta \equiv \Sigma^\infty_\Delta$ and $\Phi^*_\Delta \equiv \Phi^\infty_\Delta$.

[21] Because of the structure of signaling games, the receiver's (resp. sender's) strategies are eliminated only in odd (resp. even) rounds.

Definition 5 Δ*-rationalizability (Battigalli and Siniscalchi, 2003; Battigalli, 2006)*
A strategy pair (σ, ϕ) *is* Δ*-rationalizable if it satisfies the following conditions: (i)* $G(\sigma) \subseteq \Sigma^*_\Delta$*, and (ii)* $\phi \in \Phi^*_\Delta$*.*

Intuitively, Δ-rationalizable strategies are the strategies that survive after the iterative eliminations of strategies that are not supported as the best response when the players have beliefs consistent with restrictions Δ. We define desirable restrictions Δ as follows:

$$\Delta^S \equiv \left\{ \mu^S \in \Delta(\bar{\Phi}) \;\middle|\; \begin{array}{l} \text{(i) if } m \cap \Theta_{11} \neq \emptyset \text{ and } m \cap \Theta_{21} \neq \emptyset, \text{ then } \mu^S\left(\bar{\Phi}_2(m)\right) > 0, \\ \text{(ii) if } m \cap \Theta_{22} \neq \emptyset \text{ and } m \cap \Theta_{12} \neq \emptyset, \text{ then } \mu^S\left(\bar{\Phi}_1(m)\right) > 0 \end{array} \right\}, \tag{14}$$

$$\Delta^R \equiv \bar{\Delta}(\Theta, M). \tag{15}$$

While the receiver's beliefs are not restricted, we assume that the sender certainly believes that the receiver is "skeptical," in the sense that if she observes a message that is available to the types in both the agreement and the disagreement regions, then she never excludes possibilities of the disagreement types having sent the message. The restrictions guarantee the unique selection, like the certifiable dominance, as shown in the following theorem.

Theorem 3 *There exist restrictions* Δ *such that* U^+ *is the unique informativeness supported by PBEs constructed by the* Δ*-rationalizable strategies.*

Although Δ-rationalizability can uniquely select informativeness U^+, the desirable result is sensitive to restrictions on beliefs. For example, suppose that the players' beliefs are never restricted; that is, $\Delta^S = \Delta\left(\bar{\Phi}\right)$ and $\Delta^R = \bar{\Delta}(\Theta, M)$. Even if we do not put any restrictions on the receiver's beliefs, the receiver's rationality and the certifiability assumption eliminate strategies not included in Φ_C. However, we cannot exclude the possibility of types in the agreement region Θ_{11} (resp. Θ_{22}) pooling with the types in the disagreement region Θ_{21} (resp. Θ_{12}) because such behaviors are weakly, but not strictly, dominated. As a result, the unique selection is no longer guaranteed. In particular, restrictions to the sender's beliefs are necessary for eliminating such pooling behaviors as long as $\Theta_{12} \neq \emptyset$ and $\Theta_{21} \neq \emptyset$. Consider the scenario where only the receiver's beliefs are restricted; for example, suppose that the receiver's beliefs are *skeptical* à la Milgrom and Roberts (1986), as follows:

$$\Delta^R = \left\{ \mu^R \in \bar{\Delta}(\Theta, M) \;\middle|\; \begin{array}{l} \text{if } m \cap \Theta_{11} \neq \emptyset \text{ and } m \cap \Theta_{21} \neq \emptyset, \\ \text{then } \mu^R(\Theta_{11} \cup \Theta_{12} | m) < \mu^R(\Theta_{22} \cup \Theta_{21} | m) \end{array} \right\}. \tag{16}$$

Under this restriction, the types in Θ_{11} are never pooling with the types in Θ_{21}. However, we cannot prevent the types in Θ_{22} from pooling with the types in Θ_{12} because sending message m' such that $m' \cap \Theta_{11} \neq \emptyset$, $m' \cap \Theta_{21} \neq \emptyset$, $m' \cap \Theta_{22} \neq \emptyset$, and $m' \cap \Theta_{12} \neq \emptyset$ induces the ideal action y_2 for the types in $\Theta_{22} \cup \Theta_{12}$.[22] Thus, we have to carefully restrict beliefs for obtaining the unique selection, but there might be room for discussion about whether the appropriate restrictions are reasonable.[23]

5 Extension: Partially Certifiable States

In this section, we discuss whether the unique selection of the most informative equilibrium by CUEs can be extended to the scenario where the states are partially certifiable. In contrast with the full certification environment, there are several reasonable certification structures that induce different equilibria. Hence, we restrict our attention to the environment described by Lanzi and Mathis (2008), and show that the most informative equilibrium is uniquely selected.

The baseline model is modified as follows. We define $M \equiv \Theta$, and the set of available messages is defined by $M(\theta) \equiv \{\, m \in \Theta \mid m \leq \theta \,\}$. Notice that under this modification, the certifiability of states is limited, in the sense that while any type can certify the lower bound of his type, he cannot certify the upper bound.[24] We assume the following conditions for utility functions.

Assumption 3

(i) *Both u and v are continuous in θ for any $y \in Y$.*

(ii) *Both $u(\cdot, y_1)$ and $v(\cdot, y_1)$ are strictly decreasing in θ, and both $u(\cdot, y_2)$ and $v(\cdot, y_2)$ are strictly increasing in θ.*

(iii) *$u(0, y_1) > u(0, y_2)$ and $u(1, y_1) < u(1, y_2)$.*

(iv) *$v(0, y_1) > v(0, y_2)$ and $v(1, y_1) < v(1, y_2)$.*

[22]If either $\Theta_{12} = \emptyset$ or $\Theta_{21} = \emptyset$, then the restriction to the skeptical beliefs guarantees the unique selection, as in Proposition 3 of Battigalli (2006). The details appear in Appendix B.

[23]If we adopt an extensive-form analogous of iterative admissibility, called *prudent rationalizability* à la Heifetz et al. (2011), then we can also obtain the unique selection by the same argument. The details for the same are available from the author upon request. While Brandenburger et al. (2008) provide an epistemic foundation for iterative admissibility in complete information games, its application to incomplete information games remains an open question.

[24]While Lanzi and Mathis (2008) study a finite game, their results can be easily extended to the infinite game explored in this paper.

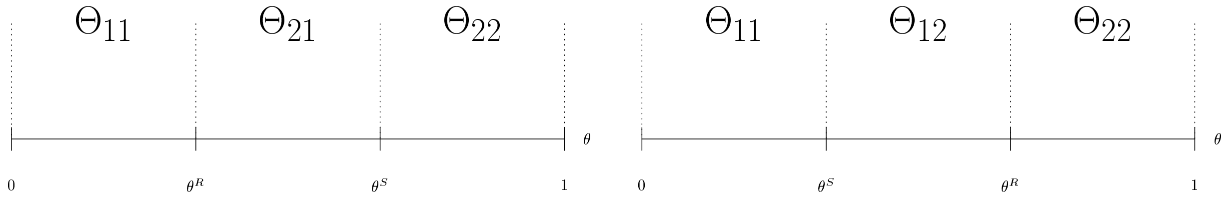

Figure 1: Distribution of Preferences: Case 1 (left) and Case 2 (right)

By Assumption 3, there exists a unique $\theta^R \in \Theta$ such that $u(\theta^R, y_1) = u(\theta^R, y_2)$, and θ^S can be defined analogously. Hence, dependent on the locations of θ^R and θ^S, either $\Theta_{12} = \emptyset$ or $\Theta_{21} = \emptyset$ holds, as shown in Figure 1. To ease reference, the former (i.e., $\theta^R < \theta^S$) and the latter (i.e., $\theta^R \geq \theta^S$) are called Case 1 and Case 2, respectively. Finally, we assume the following additional conditions to guarantee that multiple equilibria exist.

Assumption 4

(i) $\mathbb{E}[u(\theta, y_1)] \leq \mathbb{E}[u(\theta, y_2)]$.

(ii) $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] > \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$.

(iii) $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{22}] < \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{22}]$.

Except for this modification, the setup is identical to that in the baseline model. The following terminology is also introduced. We say that PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a *fully pooling equilibrium* if $\sigma^*(\theta) = 0$ for any $\theta \in \Theta$. Likewise, we say that PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a *two-partition equilibrium with cutoff* θ^* if:

$$\sigma^*(\theta) = \begin{cases} \theta^* & \text{if } \theta \geq \theta^*, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Let U^* represent the informativeness of the most informative equilibrium in this environment.

Lanzi and Mathis (2008) show that there exist multiple PBEs under Assumptions 3 and 4. In Case 1, there exist the fully pooling equilibrium and the two-partition equilibrium with cutoff θ^S. In Case 2, there exist the fully pooling equilibrium, the fully revealing equilibrium, and the two-partition equilibrium with cutoff θ^* for any $\theta^* \in [\theta^S, \theta^R)$. Furthermore, with additional assumptions, there also exist PBEs in which the equilibrium outcomes are nonmonotonic. Although there exist multiple PBEs, CUEs uniquely select the most informative equilibrium as shown in the following theorem.

Theorem 4 *Consider the partially certifiable model with Assumptions 3 and 4. Then, U^* is the unique informativeness supported by CUEs.*

Intuitively, CUEs require that the types in agreement region Θ_{22} are never pooling with the types in $\Theta_{11} \cup \Theta_{12} \cup \Theta_{21}$. In this environment, Φ_C is simplified as follows:

$$\Phi_C = \left\{ \phi \in \Phi \;\middle|\; S(\phi(m)) = \{y_2\} \text{ if } m > \theta^R \right\}. \tag{18}$$

Hence, for any type in agreement region Θ_{22}, a strategy sending message $m \leq \Theta^R$ with positive probability is certifiably dominated. As a result, only the two-partition equilibrium with cutoff θ^S and the fully revealing equilibrium can be supported as CUEs in Cases 1 and 2, respectively, and these are the most informative equilibria in the respective cases.

6 Conclusion

This paper discussed equilibrium selection in persuasion games with binary actions. There exist multiple equilibria in this environment, but the literature conventionally focus on the most informative equilibrium without formal justification. We demonstrated that the well-known existing criteria in costly signaling and cheap-talk games are useless for justifying such a convention. In particular, the announcement proofness could uniquely select the least informative equilibrium, which is incoherent with the convention. We then proposed a notion of certifiable dominance. Certifiably undominated strategies can be understood as the Δ-rationalizable strategies with additional restrictions on the sender's beliefs. We showed that the most informative equilibrium is uniquely supported by CUEs, which is consistent with the convention. It is worthwhile to emphasize that this result is irrelevant to the existence of the fully revealing equilibrium. Furthermore, the CUE could uniquely select the most informative equilibrium even if the sender's private information is partially certifiable.

While we have restricted our attention to persuasion games with binary actions so far, it is nontrivial to extend the results to the environment where the receiver has more than two actions in the following sense. First, the justification by Δ-rationalizability seems more useful than that by the certifiable dominance in this extension, but it is not clear as to what kind of additional restrictions on beliefs are needed and whether those restrictions are reasonable. Second, while Forges and Koessler (2008) provide the most comprehensive characterization of equilibrium set, because of the cost of its generality, it is hard to observe the pooling structure of each equilibrium in their characterization. Hence, the direct application of our arguments is difficult. This matter may be explored in detail in the future.

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Appendix A: Proofs

A.1 Preliminaries

First, we show Lemma 1, which is frequently used in the following proofs.

Lemma 1 *(Lemma 1 of Miura (2014))*
In any PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$, $\phi^*(m) = y^R(\theta)$ for any $\theta \in \Theta_{11} \cup \Theta_{22}$ and $m \in S(\sigma^*(\theta))$.

Proof. Suppose, in contrast, that there exists PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ with type $\theta \in \Theta_{11} \cup \Theta_{22}$ such that $\phi^*(m) \neq y^R(\theta)$ for some $m \in S(\sigma^*(\theta))$. Because of the optimality of σ^*, $\{\theta\} \notin S(\sigma^*(\theta))$. However, type θ's ideal action is $y^R(\theta)$, and it is induced by sending message $m = \{\theta\}$. That is, the sender has an incentive to deviate, which is a contradiction. ■

A.2 Proof of Proposition 1

Suppose, in contrast, that there exists PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ that is not neologism proof. That is, there exists credible neologism (d, D). By Lemma 1, $D \cap (\Theta_{11} \cup \Theta_{22} \cup \Theta_{00} \cup \Theta_{10} \cup \Theta_{20}) = \emptyset$ should hold; otherwise, Condition (N1) is violated. Next, suppose, in contrast, that $D \subseteq \Theta_{01} \cup \Theta_{02}$. Without loss of generality, assume that there exists $\theta' \in D \cap \Theta_{01}$. By construction, we can say that $BR_R(D, \mathcal{P}_A) = [0,1]$. However, for $\alpha = 0 \in BR_R(D, \mathcal{P}_A)$, $V(\theta', 0) = v(\theta', y_2) \leq \bar{V}(\theta', \phi^*(\sigma^*(\theta')))$, which contradicts Condition (N1). Thus, $D \cap (\Theta_{12} \cup \Theta_{21}) \neq \emptyset$. Furthermore, suppose, in contrast, that $D \cap \Theta_{12} \neq \emptyset$ and $D \cap \Theta_{21} \neq \emptyset$. If $BR_R(D, \mathcal{P}_A) = \{y_1\}$, then for type $\theta' \in D \cap \Theta_{12}$, Condition (N1) is violated, which is a contradiction. Likewise, we can derive a contradiction for the scenario $BR_R(D, \mathcal{P}_A) = \{y_2\}$ and $[0,1]$. Thus, we can say that either $D \cap \Theta_{12} = \emptyset$ or $D \cap \Theta_{21} = \emptyset$ should hold. We then assume that $D \cap \Theta_{12} \neq \emptyset$ and $D \cap \Theta_{21} = \emptyset$ without loss of generality, which implies that $BR_R(D, \mathcal{P}_A) = \{y_1\}$. However, for type $\theta' \in D \cap \Theta_{12}$, $V(\theta', 1) = v(\theta', y_1) \leq \bar{V}(\theta', \phi^*(\sigma^*(\theta')))$, which is a contradiction to Condition (N1). Therefore, such a credible neologism never exists. ■

A.3 Proof of Proposition 2

(i) (Existence) It is straightforward that the least informative equilibrium $(\sigma^-, \phi^-; \mathcal{P}^-)$ is the best equilibrium for the sender. Hence, by Proposition 6.1 of Matthews et al. (1991), it is strongly announcement proof.

(Uniqueness) Let $(\sigma^*, \phi^*; \mathcal{P}^*)$ be a PBE whose informativeness is $U \neq U^-$. By Lemma 1 and Assumption 1-(i), there exists subset $X \subseteq \Theta_{12} \cup \Theta_{21}$ such that $P(X) > 0$, and for any $\theta \in X$, there exists message $m \in S(\sigma^*(\theta))$ such that $\phi^*(m) \neq y^S(\theta)$. Without loss of generality, assume that $P(X \cap \Theta_{12}) > 0$. Because of the finiteness of $S(\sigma^*(\theta))$ for any θ and Assumption 2-(ii), there exists off-the-equilibrium-path message D such that (i) $D \cap (X \cap \Theta_{12}) \neq \emptyset$, (ii) $D \cup \Theta_{22} \neq \emptyset$, and (iii) $\mathbb{E}[u(\theta, y_1)|D] \leq \mathbb{E}[u(\theta, y_2)|D]$. Now, we consider the following announcement:

- The set of the deviant types is D,
- $\delta(\theta) = D$ for any $\theta \in D$, and
- $d = (\delta, D)$ and $a = (D, d)$.

Notice that $BR_R(D, \mathcal{P}_A) = \{y_2\}$. We show that this is a weakly credible announcement relative to $(\sigma^*, \phi^*; \mathcal{P}^*)$. First, because action y_2 is the ideal action for any type in D, and types in $D \cap (X \cap \Theta_{12})$ obtain the strictly higher utility under this announcement, Condition (A1) holds. Second, Condition (A2) is vacuously true because message $D \notin M(\theta)$ for any $\theta \in \Theta \backslash D$. Finally, Condition (A3) is trivially satisfied because the talking strategy is a constant function. Then, equilibrium $(\sigma^*, \phi^*; \mathcal{P}^*)$ is not strongly announcement proof.

(ii) Without Assumption 2-(ii), a PBE whose informativeness is U^- never exists. Hence, using the similar argument as in the proof of (i), we can conclude that no equilibrium is strongly announcement proof. ■

A.4 Proof of Proposition 3

($\Sigma_C \subseteq \hat{\Sigma}$) Suppose, in contrast, that there exists strategy $\sigma \in \Sigma_C$ such that $\sigma \notin \hat{\Sigma}$.

Case 1: There exists $\theta' \in \Theta_{11}$ and $m' \in S(\sigma(\theta'))$ such that $m' \not\subseteq \Theta_{11} \cup \Theta_{12} \cup \Theta_{10}$.

In this scenario, $m' \cap (\Theta_{22} \cup \Theta_{21} \cup \Theta_0 \backslash \Theta_{10}) \neq \emptyset$. Hence, there exists strategy $\phi' \in \Phi_C$ such that $\phi'(m') = y_2$. Now, we consider the following strategy $\sigma' \in \Sigma$ defined by:

$$\sigma'(\theta) \equiv \begin{cases} \{\theta'\} & \text{if } \theta = \theta', \\ \sigma(\theta) & \text{otherwise.} \end{cases} \tag{A.1}$$

Notice that $\phi(\sigma'(\theta')) = y_1 = y^S(\theta')$ holds for any $\phi \in \Phi_C$. Hence, $\bar{V}(\theta, \phi(\sigma'(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ for any $\theta \in \Theta$ and $\phi \in \Phi_C$ with strict inequality for θ' and $\phi' \in \Phi_C$, which contradicts $\sigma \in \Sigma_C$. Therefore, $m \subseteq \Theta_{11} \cup \Theta_{12} \cup \Theta_{10}$ holds for any $\theta \in \Theta_{11}$ and $m \in S(\sigma(\theta))$. Likewise, we can show that $m \subseteq \Theta_{22} \cup \Theta_{21} \cup \Theta_{20}$ holds for any $\theta \in \Theta_{22}$ and $m \in S(\sigma(\theta))$.

Case 2: There exists $\theta' \in \Theta_{12}$ and $m' \in S(\sigma(\theta'))$ such that $m' \subseteq \Theta_{11} \cup \Theta_{12} \cup \Theta_{10}$.

Notice that $\phi(m') = y_1$ for any $\phi \in \Phi_C$. Now, we consider the following strategy $\sigma' \in \Sigma$ defined by:

$$\sigma'(\theta) \equiv \begin{cases} \{\theta'\} \cup \Theta_{22} & \text{if } \theta = \theta', \\ \sigma(\theta) & \text{otherwise.} \end{cases} \tag{A.2}$$

By construction, there exists strategy $\phi' \in \Phi_C$ such that $\phi'(\sigma'(\theta')) = y_2$. Hence, $\bar{V}(\theta, \phi(\sigma'(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ holds for any $\theta \in \Theta$ and $\phi \in \Phi_C$ with strict inequality for $\theta' \in \Theta$ and $\phi' \in \Phi_C$, which contradicts $\sigma \in \Sigma_C$. Therefore, $m \cap (\Theta_{22} \cup \Theta_{21} \cup \Theta_0 \backslash \Theta_{10}) \neq \emptyset$ for any $\theta \in \Theta_{12}$ and $m \in S(\sigma(\theta))$. Likewise, we can show that $m \cap (\Theta_{11} \cup \Theta_{12} \cup \Theta_0 \backslash \Theta_{20}) \neq \emptyset$ holds for any $\theta \in \Theta_{21}$ and $m \in S(\sigma(\theta))$.

($\Sigma_C \supseteq \hat{\Sigma}$) Suppose, in contrast, that there exists strategy $\sigma \in \hat{\Sigma}$ such that $\sigma \notin \Sigma_C$. That is, there exists strategy $\sigma' \in \Sigma$ such that (i) $\bar{V}(\theta, \phi(\sigma'(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ for any $\theta \in \Theta$ and

$\phi \in \Phi_C$, and (ii) $\bar{V}(\theta', \phi'(\sigma'(\theta'))) > \bar{V}(\theta', \phi'(\sigma(\theta')))$ for some $\theta' \in \Theta$ and $\phi' \in \Phi_C$. It is obvious that $\theta' \notin \Theta_0 \backslash (\Theta_{01} \cup \Theta_{02})$ because of Condition (ii).

Case 1: $\theta' \in \Theta_{11} \cup \Theta_{22}$.

Because $\sigma \in \hat{\Sigma}$, $\phi(m) = y^S(\theta')$ holds for any $\phi \in \Phi_C$ and $m \in S(\sigma(\theta'))$, which contradicts Condition (ii).

Case 2: $\theta' \in \Theta_{12} \cup \Theta_{21}$.

Without loss of generality, we assume that $\theta' \in \Theta_{12}$. Because $\sigma \in \hat{\Sigma}$ and $\sigma \neq \sigma'$, there exists $m' \in S(\sigma(\theta'))$ such that $m' \cap (\Theta_{22} \cup \Theta_{21} \cup \Theta_0 \backslash \Theta_{10}) \neq \emptyset$ and $\sigma(m'|\theta') > \sigma'(m'|\theta')$. Also, $\theta' \in m$ for any $m \in S(\sigma'(\theta'))$. Hence, there exists strategy $\phi' \in \Phi_C$ such that (i) $\phi'(m) = y_1$ for any $m \in S(\sigma'(\theta')) \backslash \{m'\}$, and (ii) $\phi'(m') = y_2$. However, it implies that $\bar{V}(\theta', \phi'(\sigma(\theta'))) > \bar{V}(\theta', \phi'(\sigma'(\theta')))$, which contradicts Condition (i). Likewise, using the similar argument adopted here, we can derive a contradiction for the scenario where $\theta' \in \Theta_{01} \cup \Theta_{02}$.

■

A.5 Proof of Proposition 4

A.5.1 Preliminaries

Without loss of generality, assume that $\{y_1\} = \arg\max_{y \in Y} \mathbb{E}[u(\theta, y)|\Theta_{12} \cup \Theta_{21}]$. It is worthwhile to notice that if either $P(\Theta_{12}) = 0$ or $P(\Theta_{21}) = 0$, then the proposition trivially holds. Hence, hereafter, we assume that $P(\Theta_{12}) \neq \emptyset$ and $P(\Theta_{21}) \neq \emptyset$. First, we show the following lemma.

Lemma 2 *If $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a CUE, then either one of the following conditions holds:*

(i) $\phi^*(m) = y_2$ for any $\theta \in \Theta_{12}$ and $m \in S(\sigma^*(\theta))$, or

(ii) $\phi^*(m) = y_1$ for any $\theta \in \Theta_{21}$ and $m \in S(\sigma^*(\theta))$.

Proof of Lemma 2. Suppose, in contrast, that there exists CUE $(\sigma^*, \phi^*; \mathcal{P}^*)$, where there exist $\theta \in \Theta_{12}$ and $\theta' \in \Theta_{21}$ such that $\phi^*(m) \neq y_2$ and $\phi^*(m') \neq y_1$ for some $m \in S(\sigma^*(\theta))$ and $m' \in S(\sigma^*(\theta'))$. The following two cases should be checked.

Case 1: Either $\phi^*(m)$ or $\phi^*(m')$ is a degenerate distribution.

Without loss of generality, assume that $\phi^*(m) = y_1$ and $\phi^*(y_1|m') \in [0, 1)$. To hold this equilibrium, it is necessary that $\phi^*(\tilde{m}) = y_1$ for any $\tilde{m} \in M(\theta)$; otherwise, type θ never sends message m with positive probability. Hence, it implies that $\phi^*(\hat{m}) = y_1$ for $\hat{m} \equiv \{\theta, \theta'\}$.

Notice that because $\phi^*(y_1|m') < 1$ and $m' \in S(\sigma^*(\theta'))$, $\phi^*(y_1|m'') < 1$ for any $m'' \in S(\sigma^*(\theta'))$; that is, $\hat{m} \notin S(\sigma^*(\theta'))$. However, because $\hat{m} \in M(\theta')$, type θ' has an incentive to deviate from $\sigma^*(\theta')$, which is a contradiction.

Case 2: Both $\phi^*(m)$ and $\phi^*(m')$ are nondegenerate distributions.

First, notice that if there exists $\theta'' \in \Theta_{12}$ such that $\phi^*(m'') = y_1$ for some $m'' \in S(\sigma^*(\theta''))$, then we can derive a contradiction using the similar argument as in Case 1. Hence, $\phi^*(y_1|m'') \in [0, 1)$ should hold for any $\theta'' \in \Theta_{12}$ and $m'' \in S(\sigma^*(\theta''))$. Because $\sigma^* \in \Sigma_C$, it is necessary that for almost every $\theta'' \in \Theta_{12}$ and any $m'' \in S(\sigma^*(\theta''))$, there exists $\bar{\theta} \in \Theta_{21}$ such that $m'' \in S(\sigma^*(\bar{\theta}))$. Likewise, using the similar argument, we can insist that (i) $\phi^*(y_1|m''') \in (0, 1]$ for any $\theta''' \in \Theta_{21}$ and $m''' \in S(\sigma^*(\theta'''))$, and (ii) for almost every $\theta''' \in \Theta_{21}$ and any $m''' \in S(\sigma^*(\theta'''))$, there exists $\hat{\theta} \in \Theta_{12}$ such that $m''' \in S(\sigma^*(\hat{\theta}))$. Define:

$$M_D^* \equiv \left\{ m \in M \;\middle|\; \exists\theta'' \in \Theta_{12} \text{ and } \theta''' \in \Theta_{21} \text{ such that } m \in S(\sigma^*(\theta'')) \cap S(\sigma^*(\theta''')) \right\}. \quad \text{(A.3)}$$

Without loss of generality, assume that M_D^* is countable. It is worthwhile to remark that (i) $m^* \in M_D^*$ for almost every $\tilde{\theta} \in \Theta_{12} \cup \Theta_{21}$ and any $m^* \in S(\sigma^*(\tilde{\theta}))$, (ii) $m^* \notin S(\sigma^*(\tilde{\theta}))$ for any $m^* \in M_D^*$ and $\tilde{\theta} \in \Theta_{11} \cup \Theta_{22}$, and (iii) $\phi^*(y_1|m^*) \in (0, 1)$ for any $m^* \in M_D^*$. Because of Property (iii) of M_D^*, for any $m^* \in M_D^*$:

$$\mathbb{E}_{\mathcal{P}^*(m^*)}[u(\theta, y_1)]\Pr(m^*|\Theta_{12} \cup \Theta_{21}) = \mathbb{E}_{\mathcal{P}^*(m^*)}[u(\theta, y_2)]\Pr(m^*|\Theta_{12} \cup \Theta_{21}), \quad \text{(A.4)}$$

where for $T \subseteq \Theta$ and $m \in M$:

$$\Pr(m|T) \equiv \frac{1}{P(T)} \int_T \sigma^*(m|\theta) f(\theta) d\theta. \quad \text{(A.5)}$$

By summing up both sides of (A.4) for $m^* \in M_D^*$, we can obtain that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$ because of Properties (i) and (ii) of M_D^*. However, it contradicts Assumption 1-(ii). ■

A.5.1 Proof of Proposition 4

Suppose, in contrast, that there exists CUE $(\sigma^*, \phi^*; \mathcal{P}^*)$ such that $\mathbb{E}[\bar{U}(\theta, \phi^*(\sigma^*(\theta)))|\Theta_{12} \cup \Theta_{21}] \neq \mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}]$. We define $\Theta^* \equiv \{ \theta \in \Theta_{12} \cup \Theta_{21} \mid \forall m \in S(\sigma^*(\theta)),\ \phi^*(m) = y_2 \}$.

Lemma 3 $P(\Theta^*) > 0$.

Proof of Lemma 3. First, notice that if the receiver randomizes actions upon message m, then $\mathbb{E}_{\mathcal{P}^*(m)}[u(\theta, y_1)] = \mathbb{E}_{\mathcal{P}^*(m)}[u(\theta, y_2)]$ should hold. Hence, to guarantee the hypothesis, it is necessary that $P(\Theta^{**}) > 0$, where $\Theta^{**} \equiv \{ \theta \in \Theta_{12} \cup \Theta_{21} \mid \exists m \in S(\sigma^*(\theta)) \text{ such that } \phi^*(m) = y_2 \}$. It remains to show that subset $\Theta^* \subseteq \Theta^{**}$ also has a positive measure. If Condition (i) of Lemma 2 holds, then $\Theta_{12} \subseteq \Theta^*$. That is, $P(\Theta^*) > 0$ because $P(\Theta_{12}) > 0$. If Condition (ii) of Lemma 2 holds, then $\Theta^{**} \subseteq \Theta_{12}$. That is, because $y^S(\theta^{**}) = y_2$ for any $\theta^{**} \in \Theta^{**}$, if there exists message $m^{**} \in S(\sigma^*(\theta^{**}))$ such that $\phi^*(m^{**}) = y_2$, then $\phi^*(m) = y_2$ for any $m \in S(\sigma^*(\theta^{**}))$. Therefore, we can conclude that $\Theta^{**} = \Theta^*$, which implies that $P(\Theta^*) > 0$. □

The following cases are to be checked.

Case 1: $\Theta^* \subseteq \Theta_{21}$.

In this scenario, there exist $\theta \in \Theta_{12}$ and $\theta' \in \Theta^*$ such that $\phi^*(m) \neq y_2$ for some $m \in S(\sigma^*(\theta))$ and $\phi^*(m') \neq y_1$ for some $m' \in S(\sigma^*(\theta'))$, which is a contradiction to Lemma 2.

Case 2: $\Theta^* \subseteq \Theta_{12}$.

Lemma 4 *There exist $\theta \in \Theta^*$ and $\theta' \in \Theta_{21}$ such that $S(\sigma^*(\theta)) \cap S(\sigma^*(\theta')) \neq \emptyset$.*

Proof of Lemma 4. Suppose, in contrast, that for any $\theta \in \Theta^*$ and $\theta' \in \Theta_{21}$, $S(\sigma^*(\theta)) \cap S(\sigma^*(\theta')) = \emptyset$. Because $\sigma^* \in \Sigma_C$, it is necessary that for any θ^* and $m \in S(\sigma^*(\theta))$, there exists $\theta' \in \Theta_{20}$ such that $m \in S(\sigma^*(\theta'))$. We define:

$$M^* \equiv \left\{ m \in M \;\middle|\; \exists \theta \in \Theta^* \text{ and } \theta' \in \Theta_{20} \text{ such that } m \in S(\sigma^*(\theta)) \cap S(\sigma^*(\theta')) \right\}. \tag{A.6}$$

Without loss of generality, M^* is assumed to be countable. Notice that (i) $m^* \in M^*$ for any $\theta \in \Theta^*$ and $m^* \in S(\sigma^*(\theta^*))$, (ii) $m^* \notin S(\sigma^*(\theta))$ for any $m^* \in M^*$ and $\theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_{21}$, and (iii) $\phi^*(m^*) = y_2$ for any $m^* \in M^*$. Because of Property (iii) of M^*, for any $m^* \in M^*$:

$$\mathbb{E}_{\mathcal{P}^*(m^*)}[u(\theta, y_1)]\Pr(m^*|\Theta^* \cup \Theta_{20}) \leq \mathbb{E}_{\mathcal{P}^*(m^*)}[u(\theta, y_2)]\Pr(m^*|\Theta^* \cup \Theta_{20}). \tag{A.7}$$

By summing up both sides of (A.7) for any $m^* \in M^*$, we obtain that $\mathbb{E}[u(\theta, y_1)|\Theta^*] \leq \mathbb{E}[u(\theta, y_1)|\Theta^*]$ because of Properties (i), (ii), and Assumption 1-(i). However, this contradicts $\Theta^* \subseteq \Theta_{12}$. □

By Lemma 4, there exists message $m \in S(\sigma^*(\theta)) \cap S(\sigma^*(\theta'))$ for some $\theta \in \Theta^*$ and $\theta' \in \Theta_{21}$ such that $\phi^*(m) = y_2$. However, to hold equilibrium, it is necessary that $\phi^*(m') = y_2$ for any $m' \in M(\theta')$; otherwise, type θ' deviates. That is, $\theta' \in \Theta^*$, which is a contradiction to $\Theta^* \subseteq \Theta_{12}$.

Case 3: $\Theta^* \cap \Theta_{12} \neq \emptyset$ and $\Theta^* \cap \Theta_{21} \neq \emptyset$.

Because $\Theta^*_{21} \equiv \Theta^* \cap \Theta_{21} \neq \emptyset$, by Lemma 2, $\Theta_{12} \subseteq \Theta^*$ should hold. Now, suppose, in contrast, that $P(\Theta_{21} \backslash \Theta^*_{21}) > 0$. For any $\theta \in \Theta_{21} \backslash \Theta^*_{21}$, there exists message $m \in S(\sigma^*(\theta))$ such that $\phi^*(m) \neq y_2$. Hence, by the optimality of σ^*, $\phi^*(m') \neq y_2$ for any $m' \in S(\sigma^*(\theta))$. Because $\sigma^* \in \Sigma_C$ and $P(\Theta_{21} \backslash \Theta^*_{21}) > 0$, using the similar argument as that in the proof of Lemma 4, there exists message $m'' \in S(\sigma^*(\theta)) \cap S(\sigma^*(\theta'))$ for some $\theta \in \Theta_{21} \backslash \Theta^*_{21}$ and $\theta' \in \Theta_{12}$. However, because $\Theta_{12} \subseteq \Theta^*$, $\phi^*(m'') = y_2$ must hold, which is a contradiction. Therefore, we can conclude that $P(\Theta_{21} \backslash \Theta^*_{21}) = 0$, which implies that $P(\Theta^*) = P(\Theta_{12} \cup \Theta_{21})$. Now, define $\bar{\Theta}_{21} \equiv \{ \theta \in \Theta_{21} \mid \exists m \in S(\sigma^*(\theta)) \cap S(\sigma^*(\theta')) \text{ for some } \theta' \in \Theta_{22} \}$, and:

$$M^{**} \equiv \left\{ m \in M \;\middle|\; \exists \theta \in \Theta^* \backslash \bar{\Theta}_{21} \text{ such that } m \in S(\sigma^*(\theta)) \right\}. \tag{A.8}$$

Without loss of generality, assume that M^{**} is countable. Notice that (i) $m^{**} \in M^{**}$ for almost every $\theta \in \Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})$ and any $m \in S(\sigma^*(\theta))$, (ii) $m^{**} \notin S(\sigma^*(\theta))$ for any $m^{**} \in M^{**}$ and $\theta \in \Theta_{11} \cup \Theta_{22}$, and (iii) $\phi^*(m^{**}) = y_2$ for any $m^{**} \in M^{**}$. Because of Property (iii) of M^{**}, for any $m^{**} \in M^{**}$:

$$\mathbb{E}_{\mathcal{P}^*(m^{**})}[u(\theta, y_1)] \Pr(m^{**} | \Theta^* \backslash \bar{\Theta}_{21}) \leq \mathbb{E}_{\mathcal{P}^*(m^{**})}[u(\theta, y_2)] \Pr(m^{**} | \Theta^* \backslash \bar{\Theta}_{21}). \tag{A.9}$$

By summing up both sides of (A.9) for $m^{**} \in M^{**}$, we obtain $\mathbb{E}[u(\theta, y_1) | \Theta^* \backslash \bar{\Theta}_{21}] \leq \mathbb{E}[u(\theta, y_2) | \Theta^* \backslash \bar{\Theta}_{21}]$ because of Properties (i) and (ii) of M^{**}. Because $P(\Theta^* \backslash \bar{\Theta}_{21}) = P(\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21}))$, $\mathbb{E}[u(\theta, y_1) | \Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] \leq \mathbb{E}[u(\theta, y_2) | \Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]$ holds, which implies that $\mathbb{E}[u(\theta, y_1) | \Theta_{12} \cup \Theta_{21}] \leq \mathbb{E}[u(\theta, y_2) | \Theta_{12} \cup \Theta_{21}]$. However, this contradicts $\{y_1\} = \arg\max_{y \in Y} \mathbb{E}[u(\theta, y) | \Theta_{12} \cup \Theta_{21}]$. ■

A.6 Proof of Theorem 2

First, by Proposition 3, it is obvious that $\sigma^+ \in \Sigma_C$, and then U^+ is supported by a CUE. We now need to show the uniqueness. Suppose, in contrast, that there exists CUE $(\sigma^*, \phi^*; \mathcal{P}^*)$ such that $\mathbb{E}[\bar{U}(\theta, \phi^*(\sigma^*(\theta)))] \neq U^+$. By Lemma 1 and Proposition 3, $\phi^*(m) = y^R(\theta)$ holds for any $\theta \in \Theta_{11} \cup \Theta_{22}$ and $m \in S(\sigma^*(\theta))$ in any CUE. Hence, it should hold that $\mathbb{E}[\bar{U}(\theta, \phi^*(\sigma^*(\theta))) | \Theta_{12} \cup \Theta_{21}] \neq \mathbb{E}[\bar{U}(\theta, \phi^+(\sigma^+(\theta))) | \Theta_{12} \cup \Theta_{21}]$. However, it is a contradiction to Proposition 4. ■

A.7 Proof of Theorem 3

A.7.1 Preliminaries

We introduce the following additional notation:

$$\bar{\Phi}_C \equiv \bar{\Phi} \cap \Phi_C, \tag{A.10}$$

$$G_\Delta \equiv \left\{ (\theta, m) \in \Sigma^0_\Delta \;\middle|\; m \cap \left\{ \begin{array}{cl} \Theta_{21} = \emptyset & \text{if } \theta \in \Theta_{11}, \\ \Theta_{12} = \emptyset & \text{if } \theta \in \Theta_{22}, \\ \Theta_{22} \cup \Theta_{21} \cup \Theta_0 \backslash \Theta_{10} \neq \emptyset & \text{if } \theta \in \Theta_{12}, \\ \Theta_{11} \cap \Theta_{12} \cup \Theta_0 \backslash \Theta_{20} \neq \emptyset & \text{if } \theta \in \Theta_{21} \end{array} \right. \right\}, \tag{A.11}$$

$$M_\Delta \equiv \left\{ m \in M \;\middle|\; \exists \theta,\, \theta' \in m \text{ such that } y^R(\theta) \neq y^R(\theta') \right\}. \tag{A.12}$$

Proposition 5 $\Phi^1_\Delta = \bar{\Phi}_C$.

Proof. ($\Phi^1_\Delta \subseteq \bar{\Phi}_C$) Suppose, in contrast, that there exists $\phi \in \Phi^1_\Delta$ such that $\phi \notin \bar{\Phi}_C$. That is, there exists message $m \in M$ such that $\phi(m) \notin \bigcup_{\theta \in m}\{y^R(\theta)\}$. If $m \notin M_\Delta$, then $\bigcup_{\theta \in m}\{y^R(\theta)\} = Y$, which is a contradiction to $\phi(m) \in Y$. Hence, without loss of generality, we assume that $y^R(\theta) = y^R(\theta') = y_1$ for any θ, $\theta' \in m$. That is, $\phi(m) = y_2$. However, because $\Theta^1_\Delta(m) = m$ for any $m \in M$, $BR_R(m, \mu^R(m)) = \{y_1\}$ should hold for any $\mu^R \in \Delta^R$ such that $\mu^R\left(\Theta^1_\Delta(m)|m\right) = 1$, which contradicts $\phi \in \Phi^1_\Delta$.

($\Phi^1_\Delta \supseteq \bar{\Phi}_C$) We fix $\phi \in \bar{\Phi}_C$ arbitrarily. If $m \notin M_\Delta$, then we can show that $\phi \in \Phi^1_\Delta$ by the same argument used in the necessary part. We then suppose that $m \in M_\Delta$. Because $\Theta^1_\Delta(m) = m$ for any $m \in M$, there exist θ, $\theta' \in \Theta^1_\Delta(m)$ such that $y^R(\theta) = y_1$ and $y^R(\theta') = y_2$. Hence, there exist beliefs μ^R, $\mu^{R'} \in \Delta^R$ such that $\mu^R(\theta|m) = 1$ and $\mu^{R'}(\theta'|m) = 1$. Then, $BR_R(m, \mu^R(m)) = \{y_1\}$ and $BR_R(m, \mu^{R'}(m)) = \{y_2\}$ holds. Therefore, we can say that $\phi \in \Phi^1_\Delta$. ■

Proposition 6 $\Sigma^2_\Delta = G_\Delta$.

Proof. ($\Sigma^2_\Delta \subseteq G_\Delta$) Suppose, in contrast, that there exists $(\theta, m) \in \Sigma^2_\Delta$ such that $(\theta, m) \notin G_\Delta$.

Case 1: $\theta \in \Theta_{11} \cup \Theta_{22}$.

Without loss of generality, we assume that $\theta \in \Theta_{11}$. That is, $m \cap \Theta_{21} \neq \emptyset$. By Proposition 5, there exists $\phi \in \Phi^1_\Delta$ such that $\phi(m) = y_2$. Hence, for any $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right) = 1$, $\sum_{y \in Y} v(\theta, y)\pi(y|m, \mu^S) < v(\theta, y_1)$. However, if type θ sends message $m' \equiv \{\theta\}$, then $\phi(m) = y_1$ for any $\phi \in \Phi^1_\Delta$ by Proposition 5. That is, for any $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right) = 1$,

$\sum_{y\in Y} v(\theta,y)\pi(y|m',\mu^S) = v(\theta,y_1)$ holds, which is a contradiction to $(\theta,m)\in\Sigma^2_\Delta$. Likewise, we can derive a contradiction for $\theta\in\Theta_{22}$ by the similar argument.

Case 2: $\theta\in\Theta_{12}\cup\Theta_{21}$.

Without loss of generality, we assume that $\theta\in\Theta_{12}$. That is, $m\subseteq\Theta_{11}\cup\Theta_{12}\cup\Theta_{10}$. By Proposition 5, $\phi(m)=y_1$ holds for any $\phi\in\Phi^1_\Delta$. Hence, for any $\mu^S\in\Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right)=1$, $\sum_{y\in Y} v(\theta,y)\pi(y|m,\mu^S)=v(\theta,y_1)$. However, suppose that type θ sends message $m'\in M(\theta)$ such that $m'\cap\Theta_{11}\neq\emptyset$ and $m'\cap\Theta_{21}\neq\emptyset$. By Proposition 5, there exists $\phi\in\Phi^1_\Delta$ such that $\phi(m')=y_2$. Hence, for any $\mu^S\in\Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right)=1$, $\sum_{y\in Y} v(\theta,y)\pi(y|m',\mu^S) > v(\theta,y_1)$, which is a contradiction to $(\theta,m)\in\Sigma^2_\Delta$. Likewise, we can derive a contradiction for $\theta\in\Theta_{21}$ using the similar argument.

($\Sigma^2_\Delta\supseteq G_\Delta$) We fix $(\theta,m)\in G_\Delta$ arbitrarily. The following cases should be checked.

Case 1: $\theta\in\Theta_{11}\cup\Theta_{22}$.

Without loss of generality, assume that $\theta\in\Theta_{11}$. That is, $m\cap\Theta_{21}=\emptyset$ and $y^S(\theta)=y_1$. First, suppose that $m\notin M_\Delta$; that is, $y^R(\theta)=y_1$ for any $\theta\in m$. By Proposition 5, $\phi(m)=y_1$ for any $\phi\in\Phi^1_\Delta$. Then, for any $\mu^S\in\Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right)=1$, $\sum_{y\in Y} v(\theta,y)\pi(y|m,\mu^S)=v(\theta,y_1)$. Thus, $m\in BR_S(\theta,\mu^S)$ holds. Next, suppose that $m\in M_\Delta$. Because of $m\cap\Theta_{11}\neq\emptyset$ and Proposition 5, there exists $\phi'\in\Phi^1_\Delta$ such that $\phi'(m)=y_1$. Because $m\cap\Theta_{21}=\emptyset$, there exists $\mu^{S'}\in\Delta^S$ such that $\mu^{S'}(\phi')=1$ and $\sum_{y\in Y} v(\theta,y)\pi(y|m,\mu^{S'})=v(\theta,y_1)$; that is, $m\in BR_S(\theta,\mu^{S'})$ holds. Likewise, we can show the case for $\theta\in\Theta_{22}$ by the similar argument.

Case 2: $\theta\in\Theta_{12}\cup\Theta_{21}$.

Without loss of generality, we assume that $\theta\in\Theta_{12}$. That is, $m\cap(\Theta_{22}\cup\Theta_{21}\cup\Theta_0\backslash\Theta_{10})\neq\emptyset$, and then, $m\in M_\Delta$. Hence, by Proposition 5, there exist ϕ', $\phi''\in\Phi^1_\Delta$ such that for any $m'\in M_\Delta$:

$$\phi'(m')=\begin{cases} y_1 & \text{if } m'\cap\Theta_{22}\neq\emptyset,\\ y_2 & \text{otherwise,}\end{cases} \tag{A.13}$$

$$\phi''(m')=\begin{cases} y_2 & \text{if } m\cap\Theta_{22}\neq\emptyset,\\ y_1 & \text{otherwise.}\end{cases} \tag{A.14}$$

First, suppose that $m \cap \Theta_{22} \neq \emptyset$. Consider belief $\mu^{S'} \in \Delta^S$ such that:

$$\mu^{S'}(\phi) = \begin{cases} 1/3 & \text{if } \phi = \phi', \\ 2/3 & \text{if } \phi = \phi''. \end{cases} \tag{A.15}$$

It is then obvious that $m \in BR_S(\theta, \mu^{S'})$. Next, suppose that $m \cap \Theta_{22} = \emptyset$. Consider the following belief $\mu^{S''} \in \Delta^S$ such that:

$$\mu^{S''}(\phi) = \begin{cases} 2/3 & \text{if } \phi = \phi', \\ 1/3 & \text{if } \phi = \phi''. \end{cases} \tag{A.16}$$

Again, it is obvious that $m \in BR_S(\theta, \mu^{S''})$. Likewise, we can show the scenario for $\theta \in \Theta_{21}$.

Case 3: $\theta \in \Theta_0 \backslash (\Theta_{01} \cup \Theta_{02})$.

In this case, actions y_1 and y_2 are indifferent for type θ. Hence, it is obvious that $m \in BR_S(\theta, \mu^S)$ holds for any $m \in M(\theta)$ and $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right) = 1$.

Case 4: $\theta \in \Theta_{01} \cup \Theta_{02}$.

Without loss of generality, we assume that $\theta \in \Theta_{01}$. Notice that $m' \in M_\Delta$ holds for any $m' \in M(\theta)$. First, suppose that $m \cap \Theta_{22} \neq \emptyset$ and $m \cap \Theta_{12} \neq \emptyset$. By Proposition 5, there exist ϕ', $\phi'' \in \Phi^1_\Delta$ such that for any $m' \in M_\Delta$:

$$\phi'(m') = \begin{cases} y_1 & \text{if } m' \cap \Theta_{22} \neq \emptyset \text{ and } m' \cap \Theta_{12} \neq \emptyset, \\ y_2 & \text{otherwise}, \end{cases} \tag{A.17}$$

$$\phi''(m') = \begin{cases} y_2 & \text{if } m' \cap \Theta_{22} \neq \emptyset \text{ and } m' \cap \Theta_{12} \neq \emptyset, \\ y_1 & \text{otherwise}. \end{cases} \tag{A.18}$$

Consider belief $\mu^{S'} \in \Delta^S$ such that:

$$\mu^{S'}(\phi) = \begin{cases} 2/3 & \text{if } \phi = \phi', \\ 1/3 & \text{if } \phi = \phi''. \end{cases} \tag{A.19}$$

It is then obvious that $m \in BR_S(\theta, \mu^{S'})$. Next, suppose that $[m \cap \Theta_{22} = \emptyset$ or $m \cap \Theta_{12} = \emptyset]$, $m \cap \Theta_{11} \neq \emptyset$, and $m \cap \Theta_{21} \neq \emptyset$. By Proposition 5, there exist $\hat{\phi}$, $\tilde{\phi} \in \Phi^1_\Delta$ such that for any

$m' \in M_\Delta$:

$$\hat{\phi}(m') = \begin{cases} y_1 & \text{if } [m' \cap \Theta_{22} = \emptyset \text{ or } m' \cap \Theta_{12} = \emptyset], m' \cap \Theta_{11} \neq \emptyset, \text{ and } m' \cap \Theta_{21} \neq \emptyset, \\ y_2 & \text{otherwise.} \end{cases} \tag{A.20}$$

$$\tilde{\phi}(m') = \begin{cases} y_2 & \text{if } [m' \cap \Theta_{22} = \emptyset \text{ or } m' \cap \Theta_{12} = \emptyset], m' \cap \Theta_{11} \neq \emptyset, \text{ and } m' \cap \Theta_{21} \neq \emptyset, \\ y_1 & \text{otherwise.} \end{cases} \tag{A.21}$$

Then, for belief $\mu^{S''} \in \Delta_S$ such that:

$$\mu^{S''}(\phi) = \begin{cases} 2/3 & \text{if } \phi = \hat{\phi}, \\ 1/3 & \text{if } \phi = \tilde{\phi}, \end{cases} \tag{A.22}$$

$m \in BR_S(\theta, \mu^{S''})$ holds. Finally, suppose that $[m \cap \Theta_{22} = \emptyset$ or $m \cap \Theta_{12} = \emptyset]$ and $[m \cap \Theta_{11} = \emptyset$ or $m \cap \Theta_{21} = \emptyset]$. Because $m \in M(\theta)$, there exists $\phi''' \in \Phi^1_\Delta$ such that $\phi'''(m) = y_1$. Then, under belief $\mu^{S'''} \in \Delta^S$ such that $\mu^{S'''}(\phi''') = 1$, $m \in BR_S(\theta, \mu^{S'''})$ holds.

Therefore, we can conclude $\Sigma^2_\Delta = G_\Delta$. ■

A.8 Proof of Theorem 3

(Existence) Define:

$$\sigma^*(\theta) \equiv \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0, \\ \Theta_{12} \cup \Theta_{21} & \text{otherwise} \end{cases}, \tag{A.23}$$

$$\Phi^* \equiv \left\{ \phi^* \in \bar{\Sigma} \;\middle|\; \phi^*(m) = \begin{cases} y^R(\theta) & \text{if } m = \{\theta\}, \\ y_1 & \text{if } m \cap \Theta_{12} \neq \emptyset \end{cases} \right\}, \tag{A.24}$$

$$\Phi^{**} \equiv \left\{ \phi^{**} \in \bar{\Sigma} \;\middle|\; \phi^{**}(m) = \begin{cases} y^R(\theta) & \text{if } m = \{\theta\}, \\ y_2 & \text{if } m \cap \Theta_{21} \neq \emptyset \end{cases} \right\} \tag{A.25}$$

It is straightforward that for any $\phi^* \in \Phi^*$, (σ^*, ϕ^*) constructs a PBE whose informativeness is U^+. Hence, it remains to show that (σ^*, ϕ^*) is Δ-rationalizable. That is, it is sufficient to show that $G(\sigma^*) \subseteq \Sigma^{2k}_\Delta$, $\Phi^* \cap \Phi^{2k-1}_\Delta \neq \emptyset$, and $\Phi^{**} \cap \Phi^{2k-1}_\Delta \neq \emptyset$ for any $k \geq 0$. We show the statement by induction. If $k = 0$, then the statement is obvious because $\Sigma^0_\Delta = \bigcup_{\sigma \in \bar{\Sigma}} G(\sigma)$ and $\Phi^{-1}_\Delta = \bar{\Phi}$. We then suppose that $G(\sigma^*) \subseteq \Sigma^{2k}_\Delta$, $\Phi^* \cap \Phi^{2k-1}_\Delta \neq \emptyset$, and $\Phi^{**} \cap \Phi^{2k-1}_\Delta \neq \emptyset$ for some $k > 0$ as an induction hypothesis.

First, we show that $\Phi^* \cap \Phi_\Delta^{2k+1} \neq \emptyset$. Suppose, in contrast, that $\Phi^* \cap \Phi_\Delta^{2k+1} = \emptyset$. That is, for any $\phi \in \Phi_\Delta^{2k+1}$, either one of the following holds: (i) there exists message $m' = \{\theta\}$ such that $\phi(m') \neq y^R(\theta)$, or (ii) there exists message m'' such that $m'' \cap \Theta_{12} \neq \emptyset$ and $\phi(m'') = y_2$. We fix $\phi \in \Phi_\Delta^{2k+1}$ arbitrarily. The following cases need be checked.

Case 1: There exists message $m' = \{\theta\}$ such that $\phi(m') \neq y^R(\theta)$.

By Proposition 5, $\phi \notin \Phi_\Delta^1$ holds. However, it implies that $\phi \notin \Phi_\Delta^{2k+1}$ because $\Phi_\Delta^{2k+1} \subseteq \Phi_\Delta^1$, which is a contradiction.

Case 2: There exists message m'' such that $m'' \cap \Theta_{12} \neq \emptyset$ and $\phi(m'') = y_2$.

First, suppose that $\Theta_\Delta^{2k+1}(m'') \neq \emptyset$. We define:

$$M'' \equiv \left\{ m'' \in M \;\middle|\; \text{(i) } \Theta_\Delta^{2k+1}(m'') \neq \emptyset, \text{ (ii) } m'' \cap \Theta_{12} \neq \emptyset, \text{ and (iii) } \phi(m'') = y_2 \right\} \tag{A.26}$$

Now, suppose, in contrast, that for any $m'' \in M''$, there exist $\theta'' \in \Theta_\Delta^{2k+1}(m'') \cap \Theta_{12}$. For belief $\tilde{\mu}^R \in \Delta^R$ such that $\tilde{\mu}^R(\theta''|m'') = 1$ for any $m'' \in M''$, $y_1 \in BR_R(m'', \tilde{\mu}^R(m''))$ holds. Then, we define:

$$\tilde{\phi}(m) \equiv \begin{cases} y_1 & \text{if } m \in M'', \\ \phi(m) & \text{otherwise.} \end{cases} \tag{A.27}$$

By construction, $\tilde{\phi}$ is never eliminated in the previous rounds, and then $\tilde{\phi} \in \Phi^* \cap \Phi_\Delta^{2k+1}$, which is a contradiction. Thus, there should exist $\tilde{m} \in M''$ such that $\Theta_\Delta^{2k+1}(\tilde{m}) \cap \Theta_{12} = \emptyset$. That is, $(\tilde{\theta}, \tilde{m})$ is eliminated in round $l \in \{4, 6, \ldots, 2k\}$ where $\tilde{\theta} \in \tilde{m} \cap \Theta_{12}$. In other words, there must exist message $\hat{m} \in M(\tilde{\theta})$ such that for any $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi_\Delta^{l-1}\right) = 1$, $\sum_{y \in Y} v(\tilde{\theta}, y)\pi(y|\hat{m}, \mu^S) > \sum_{y \in Y} v(\tilde{\theta}, y)\pi(y|\tilde{m}, \mu^S)$. Notice that because $\Phi_\Delta^{2k+1} \subseteq \Phi_\Delta^{l-1}$, $\phi \in \Phi_\Delta^{l-1}$. On the other hand, because $\tilde{\theta} \in \Theta_\Delta^{l-1}(\tilde{m})$, there exists $\phi' \in \Phi_\Delta^{l-1}$ such that $\phi'(\tilde{m}) = y_1$. If $\tilde{m} \cap \Theta_{22} = \emptyset$, then there exists belief $\mu^S \in \Delta^S$ such that $\mu^S(\phi) = 1$, which contradicts the elimination of $(\tilde{\theta}, \tilde{m})$ in round l. Hence, $\tilde{m} \cap \Theta_{22} \neq \emptyset$ should hold. Now, consider belief $\mu_\varepsilon^S \in \Delta^S$ such that for $\varepsilon \in (0, 1]$:

$$\mu_\varepsilon^S(\tilde{\phi}) = \begin{cases} 1 - \varepsilon & \text{if } \tilde{\phi} = \phi, \\ \varepsilon & \text{if } \tilde{\phi} = \phi'. \end{cases} \tag{A.28}$$

Because $\sum_{y \in Y} v(\tilde{\theta}, y)\pi(y|\tilde{m}, \mu_\varepsilon^S)$ converges to $v(\tilde{\theta}, y_2)$ as $\varepsilon \to 0$, to eliminate $(\tilde{\theta}, \tilde{m})$ in round

l, there must exist $\hat{m} \in M(\tilde{\theta})$ such that $\phi(\hat{m}) = y_2$ for any $\phi \in \Phi_\Delta^{l-1}$. Therefore, we can insist that $\Theta_\Delta^{l-1}(\hat{m}) \cap \Theta_{12} = \emptyset$; otherwise, there exists $\phi''' \in \Phi_\Delta^{l-1}$ such that $\phi'''(\hat{m}) = y_1$. That is, $(\tilde{\theta}, \hat{m})$ is eliminated in round $l' \leq l-2$. By repeatedly applying the above argument, we can claim that there must exist $m^* \in M(\tilde{\theta})$ such that $\phi(m^*) = y_2$ for any $\phi \in \Phi_\Delta^1$. However, such m^* never exists by Proposition 5, which is a contradiction. Next, suppose that $\Theta_\Delta^{2k+1}(m'') = \emptyset$. That is, it is obvious that $\Theta_\Delta^{2k+1}(m'') \cap \Theta_{12} = \emptyset$. Hence, by the similar argument used above, we can derive a contradiction.

Therefore, we can conclude that $\Phi^* \cap \Phi_\Delta^{2k+1} \neq \emptyset$. Likewise, we can show that $\Phi^{**} \cap \Phi_\Delta^{2k+1} \neq \emptyset$.

Next, we show that $G(\sigma^*) \subseteq \Sigma_\Delta^{2k+2}$. The following two cases need to be checked.

Case 1: $\theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0$.

Because $\Phi_\Delta^{2k+1} \subseteq \Phi_\Delta^1$, $\phi(\{\theta\}) = y^R(\theta)$ must hold for any $\phi \in \Phi_\Delta^{2k+1}$ by Proposition 5. That is, for any $\theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0$, there exists belief $\mu^S \in \Delta^S$ such that $\{\theta\} \in BR_S(\theta, \mu^S)$ and $\mu^S\left(\Phi_\Delta^{2k+1}\right) = 1$.

Case 2: $\theta \in \Theta_{12} \cup \Theta_{21}$.

Without loss of generality, assume that $\theta \in \Theta_{12}$. Because $\Phi^{**} \cap \Phi_\Delta^{2k+1} \neq \emptyset$, there exists $\phi^{**} \in \Phi^{**} \cap \Phi_\Delta^{2k+1}$ such that $\phi^{**}(\Theta_{12} \cup \Theta_{21}) = y_2$. Hence, there exists belief $\mu^{S'} \in \Delta^S$ such that $\mu^{S'}(\phi^{**}) = 1$. Given $\mu^{S'}$, $\sum_{y \in Y} v(\theta, y)\pi(y|\Theta_{12} \cup \Theta_{21}, \mu^{S'}) = v(\theta, y_2) = v(\theta, y^S(\theta))$. That is, $(\Theta_{12} \cup \Theta_{21}) \in BR_S(\theta, \mu^{S'})$ holds. Likewise, we can show the scenario for $\theta \in \Theta_{21}$ by the similar argument.

Therefore, $G(\sigma^*) \subseteq \Sigma_\Delta^{2k+1}$ holds. Thus, we can conclude that (σ^*, ϕ^*) is Δ-rationalizable for any $\phi^* \in \Phi^*$.

(Uniqueness) Because of Propositions 5 and 6, we can insist that (i) $\Phi_\Delta^1 = \Phi_C$, and (ii) any type in Θ_{11} (resp. Θ_{22}) is never pooling with types in Θ_{21} (resp. Θ_{12}). Hence, we can show the uniqueness through the argument used in Proposition 4 and Theorem 2. ■

A.9 Proof of Theorem 4

We can show the statement by the similar argument used in Propositions 3, 4, and Theorem 2. Hence, it is omitted.[25] ■

[25] The details appear in Appendix B.

Appendix B: Supplementary Materials

B.1 Other existing selection criteria

We discuss the validity of the neologism proofness and the announcement proofness in the body of the paper. In this section, on the other hand, we adopt the other selection criteria that are well known in the literature. As mentioned above, those criteria do not derive the consistent results with the convention focusing on the most informative equilibrium. To compare the results in the body of the paper, Assumptions 1 and 2 are maintained throughout this subsection. We introduce the following additional notation. Let $BR_R(m)$ be the set of the receiver's best responses to message m under some posterior satisfying Requirement 1 defined by:

$$BR_R(m) \equiv \bigcup_{\mathcal{P}:\int_m \mathcal{P}(\theta|m)d\theta=1} BR_R(m,\mathcal{P}). \tag{B.1}$$

For subset $n \subseteq m$, let $B(n,m)$ be the set of the receiver's best responses to message m when her belief is concentrated on subset n defined by:

$$BR_R(n,m) \equiv \bigcup_{\mathcal{P}:\int_n \mathcal{P}(\theta|m)d\theta=1} BR_R(m,\mathcal{P}). \tag{B.2}$$

Define $X(\sigma^*,\phi^*;\mathcal{P}^*) \equiv \left\{ \theta \in \Theta_{12} \cup \Theta_{21} \;\middle|\; \phi^*(\sigma^*(\theta)) = y^R(\theta) \right\}$ given a pure-strategy PBE $(\sigma^*,\phi^*;\mathcal{P}^*)$. This is a set of states lying in the disagreement regions where the receiver undertakes her ideal action $y^R(\theta)$ in equilibrium. The following lemma is a useful result frequently used hereafter.

Lemma 5 *(Lemma 2 of Miura (2014))*
If there exists a pure strategy PBE $(\sigma^*,\phi^*;\mathcal{P}^*)$ such that $X(\sigma^*,\phi^*;\mathcal{P}^*) \neq \emptyset$, then either $X(\sigma^*,\phi^*;\mathcal{P}^*) \subseteq \Theta_{12}$ or $X(\sigma^*,\phi^*;\mathcal{P}^*) \subseteq \Theta_{21}$.

Proof. If either $\Theta_{12} = \emptyset$ or $\Theta_{21} = \emptyset$, then the statement trivial. Hence, we assume that $\Theta_{12} \neq \emptyset$ and $\Theta_{21} \neq \emptyset$. Suppose, in contrast, that there exists pure-strategy PBE $(\sigma^*,\phi^*;\mathcal{P}^*)$ such that $X(\sigma^*,\phi^*;\mathcal{P}^*) \cap \Theta_{12} \neq \emptyset$ and $X(\sigma^*,\phi^*;\mathcal{P}^*) \cap \Theta_{21} \neq \emptyset$. Choose $\theta \in X(\sigma^*,\phi^*;\mathcal{P}^*) \cap \Theta_{12}$ and $\theta' \in X(\sigma^*,\phi^*;\mathcal{P}^*) \cap \Theta_{21}$, arbitrarily. However, there is no incentive-compatible reaction to message $m = \{\theta,\theta'\} \in M(\theta) \cap M(\theta')$; if $\phi^*(\{\theta,\theta'\}) = y_2$, then type θ has an incentive to deviate from $\sigma^*(\theta)$, and if $\phi^*(\{\theta,\theta'\}) = y_1$, then type θ' has an incentive to deviate, which is a contradiction. ■

B.1.1 Intuitive criterion

The intuitive criterion, one of the most well-known criteria, has no bite in our environment like cheap-talk games. However, the reason is different from that of cheap-talk games. Because any type can send any message without costs in cheap-talk games, for any equilibrium, there exists an outcome-equivalent equilibrium in which all messages are used on the equilibrium path. On the other hand, the message set varies dependent on the type in persuasion games, such argument cannot directly applied. Let $J(m) \equiv \left\{ \theta \in m \mid \bar{V}(\theta, \phi^*(\sigma^*(\theta))) > \max_{\alpha \in BR_R(m)} V(\theta, \alpha) \right\}$ be the set of types who never deviate to off-the-equilibrium-path message m under equilibrium $(\sigma^*, \phi^*; \mathcal{P}^*)$.

Definition 6 *Intuitive Criterion (Cho and Kreps, 1987)*
An equilibrium $(\sigma^*, \phi^*; \mathcal{P}^*)$ fails the intuitive criterion if there exists an off-the-equilibrium-path message m and type $\theta \in m$ such that:

$$\bar{V}(\theta, \phi^*(\sigma^*(\theta))) < \min_{\alpha \in BR_R(m \backslash J(m), m)} V(\theta, \alpha). \tag{B.3}$$

Proposition 7 *Any PBE satisfies the intuitive criterion.*

Proof. Suppose, in contrast, that there exists PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ that fails the intuitive criterion. That is, there exists off-the-equilibrium-path message m' and state $\theta' \in m'$ satisfying (B.3). By Lemma 1, $\theta' \in \Theta_{12} \cup \Theta_{21} \cup \Theta_{01} \cup \Theta_{02}$ should hold. Suppose, in contrast, that $\theta' \in \Theta_{01} \cup \Theta_{02}$. Without loss of generality, we assume that $\theta' \in \Theta_{01}$. Because $\theta' \in m'$, y_1, $y_2 \in BR_R(m')$ holds. That is, for any $\theta \in m'$, $\bar{V}(\theta, \phi^*(\sigma^*(\theta))) \leq \max_{\alpha \in BR_R(m')} V(\theta, \alpha) = v(\theta, y^S(\theta))$, which means that $J(m') = \emptyset$. However, this implies that $\bar{V}(\theta', \phi^*(\sigma^*(\theta'))) \geq v(\theta', y_2) = \min_{\alpha \in BR_R(m' \backslash J(m'), m')} V(\theta', \alpha)$, which contradicts (B.3). Thus, $\theta' \in \Theta_{12} \cup \Theta_{21}$, and then assume that $\theta' \in \Theta_{12}$ without loss of generality. To satisfy (B.3), $m' \backslash J(m') \subseteq \Theta_{21} \cup \Theta_{22} \cup \Theta_{20}$ should hold; otherwise, $y_1 \in BR_R(m' \backslash J(m'), m')$, and then (B.3) is violated. That is, $\theta' \in J(m')$ holds. However, because $m' \cap (\Theta_{21} \cup \Theta_{22} \cup \Theta_{20}) \neq \emptyset$, $y_2 \in BR_R(m')$, which implies that $\max_{\alpha \in BR_R(m')} V(\theta', \alpha) = v(\theta', y_2) \geq \bar{V}(\theta', \phi^*(\sigma^*(\theta')))$. That is, $\theta' \notin J(m')$, which is a contradiction. ■

B.1.2 D1 and D2 criteria

Like the intuitive criterion, D1 and D2 criteria have no bite. Formally, any informativeness between $[U^-, U^+]$ can be supported by some D1 and D2 equilibria. For PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$, define:

$$D(\theta, m) \equiv \left\{ \alpha \in BR_R(m) \;\middle|\; \bar{V}(\theta, \phi^*(\sigma^*(\theta))) < V(\theta, \alpha) \right\}, \tag{B.4}$$

$$D^0(\theta, m) \equiv \left\{ \alpha \in BR_R(m) \;\middle|\; \bar{V}(\theta, \phi^*(\sigma^*(\theta))) = V(\theta, \alpha) \right\}, \tag{B.5}$$

which are the sets of best responses to off-the-equilibrium-path message m such that type θ strictly prefers and indifferent to the equilibrium outcome, respectively.[26]

Definition 7 *D1 and D2 Criteria (Cho and Kreps, 1987)*

(i) An equilibrium belief $\mathcal{P}^*$ satisfies the D1 criterion if the following condition holds: for any off-the-equilibrium-path message m, if there exist types $\theta, \theta' \in m$ such that:

$$D(\theta, m) \cup D^0(\theta, m) \subsetneq D(\theta', m), \tag{B.6}$$

then $\theta \notin S(\mathcal{P}^*(m))$.

(ii) An equilibrium belief $\mathcal{P}^*$ satisfies the D2 criterion if the following condition holds: for any off-the-equilibrium-path message m, if there exists type $\theta \in m$ such that:

$$D(\theta, m) \cup D^0(\theta, m) \subsetneq \bigcup_{\theta' \in M^{-1}(m) \setminus \{\theta\}} D(\theta', m), \tag{B.7}$$

then $\theta \notin S(\mathcal{P}^*(m))$.[27]

(iii) A PBE supported by a belief satisfying the D1 (resp. D2) criterion is called D1 (resp. D2) equilibrium.

Proposition 8 *For any informativeness $U \in [U^-, U^+]$, there exists D1 and D2 equilibria whose informativeness is U.*

Proof. It is sufficient to show that for any $U \in [U^-, U^+]$, there exists a pure strategy D2 equilibrium whose informativeness is U. For $U = U^-$, we consider PBE $(\sigma^-; \mu^-; \mathcal{P}^-)$ characterized in Theorem

[26] If $D(\theta, m) = \{0\}$ or $\{1\}$, then we simply represent it by $D(\theta, m) = \{y_1\}$ or $\{y_2\}$ with some abuse of notation. It is same for $D^0(\theta, m)$.

[27] Cho and Sobel (1990) call this criterion the universal divinity, which is slightly different from the original definition by Banks and Sobel (1987).

1. By construction, it is straightforward that $\mathcal{P}^-$ satisfies the D2 criterion because $D(\theta, m) = \emptyset$ for any $\theta \in \Theta$ and $m \in M(\theta)$. Hence, suppose that $U \neq U^-$. By Theorem 1 and Lemma 5, it is sufficient to focus on a pure-strategy PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ whose informativeness is U such that (i) $P\left(X(\sigma^*; \phi^*; \mathcal{P}^*)\right) > 0$ and $X(\sigma^*, \phi^*; \mathcal{P}^*) \subseteq \Theta_{12}$, and (ii) for any off-the-equilibrium-path message m such that $m \cap X(\sigma^*, \phi^*; \mathcal{P}^*) \neq \emptyset$, $S\left(\mathcal{P}^*(m)\right) \subseteq m \cap X(\sigma^*, \phi^*; \mathcal{P}^*)$. Notice that, by construction, any types in $\Theta \backslash X(\sigma^*, \phi^*; \mathcal{P}^*)$ obtain the highest utility in this equilibrium, and then never deviate under any belief. Hence, it reminds to show that (B.7) never holds for arbitrary fixed off-the-equilibrium-path message $m' \in \bigcup_{\theta \in X(\sigma^*, \phi^*; \mathcal{P}^*)} M(\theta)$ and $\theta' \in S(\mathcal{P}^*(m'))$. First, suppose that $m' \subseteq (\Theta_{11} \cup \Theta_{12} \cup \Theta_{10})$. In this scenario, $BR_R(m') = \{y_1\}$, and then $D(\theta', m') \cup D^0(\theta', m') = \{y_1\}$ holds. Also, for any $\theta \in M^{-1}(m') \backslash \{\theta'\}$, $D(\theta, m') = \emptyset$. Thus, (B.7) is violated. Next, suppose that $m' \cap (\Theta_{22} \cup \Theta_{21} \cup (\Theta_0 \backslash \Theta_{10})) \neq \emptyset$. In this scenario, $BR_R(m') = [0, 1]$, and then $D(\theta', m') \cup D^0(\theta', m') = [0, 1]$. That is, (B.7) is never satisfied. Therefore, $\mathcal{P}^*$ satisfies the D2 criterion. ■

B.1.3 Undefeatedness

So far, we have applied criteria that likely select informative equilibrium, i.e., separating equilibrium, in the context of costly signaling games. On the other hand, the defeatedness likely selects uninformative equilibrium. However, this criterion also has no power in our environment as follows.[28] In this subsection, we focus on pure strategy equilibria following Mailath et al. (1993).

Definition 8 *Undefeated Equilibrium (Mailath et al., 1993)*

(i) PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ defeats PBE $(\sigma', \phi'; \mathcal{P}')$ if there exists message $m \in M$ such that:

(U1) $\sigma'(\theta) \neq m$ for any $\theta \in \Theta$, and $K \equiv \{\, \theta \in \Theta \mid \sigma^*(\theta) = m \,\} \neq \emptyset$;

(U2) $v(\theta, \phi^*(\sigma^*(\theta))) \geq v(\theta, \phi'(\sigma'(\theta)))$ for any $\theta \in K$ with strictly inequality for some $\theta' \in K$; and

(U3) there exists type $\hat{\theta} \in K$ such that:

$$\mathcal{P}'(\hat{\theta}|m) \neq \frac{f(\hat{\theta})\pi(\hat{\theta})}{\int_\Theta f(\theta)\pi(\theta)d\theta} \tag{B.8}$$

for any $\pi : \Theta \to [0, 1]$ satisfying:

[28] In contrast, Celik (2014) considers a model where a privately informed seller strategically discloses the information about the quality of a product to an uninformed buyer, and successfully selects the unique equilibrium by applying the undefeatedness. The difference from our environment is that (i) the buyer also has private information about her own preference, and (ii) the seller strategically sets the price of the product.

- *if* $\theta \in K$ *and* $v(\theta, \phi^*(\sigma^*(\theta))) > v(\theta, \phi'(\sigma'(\theta)))$*, then* $\pi(\theta) = 1$*; and*
- *if* $\theta \notin K$*, then* $\pi(\theta) = 0$*.*

(ii) PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ *is undefeated if no PBE defeats* $(\sigma^*, \phi^*; \mathcal{P}^*)$*.*

Proposition 9 *For any informativeness* $U \in [U^-, U^+]$*, there exists an undefeated equilibrium* $(\sigma^*, \phi^*; \mathcal{P}^*)$ *whose informativeness is* U*.*

Proof. Suppose, in contrast, that there exists $U \in [U^-, U^+]$ such that any PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ whose informativeness is U is not undefeated. Now, we focus on the following PBE $(\sigma^* \phi^*; \mathcal{P}^*)$:

- $P(X(\sigma^*, \phi^*; \mathcal{P}^*)) > 0$ and $X(\sigma^*, \phi^*; \mathcal{P}^*) \subseteq \Theta_{12}$,
- $\phi^*(\sigma^*(\theta)) = y^S(\theta)$ for any $\theta \in \Theta_{01} \cup \Theta_{02}$,
- $\mathbb{E}[u(\theta, \phi^*(\sigma^*(\theta)))] = U$, and
- for any off-the-equilibrium-path message m such that $m \cap X(\sigma^*, \phi^*; \mathcal{P}^*) \neq \emptyset$, $S(\mathcal{P}^*(m)) \subseteq m \cap X(\sigma^*, \phi^*; \mathcal{P}^*)$.

By the hypothesis, there exists another PBE $(\sigma', \phi'; \mathcal{P}')$ that defeats PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$; that is, there exists a defeating message m' satisfying Conditions (U1) - (U3). Define $K' \equiv \{ \theta \in \Theta \mid \sigma'(\theta) = m' \} \neq \emptyset$. Because at least one type in K' should be strictly improved in equilibrium $(\sigma', \phi'; \mathcal{P}')$, $K' \cap X(\sigma^*, \phi^*; \mathcal{P}^*) \neq \emptyset$, $m' \in \bigcup_{\theta \in X(\sigma^*, \phi^*; \mathcal{P}^*)} M(\theta)$, and $\phi'(m') = y_2$. Notice that because $K' \subseteq m'$, the following belief under message m' is consistent with PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$:

$$\mathcal{P}^*(\theta | m') = \begin{cases} \frac{f(\theta)}{\int_{K' \cap X(\sigma^*, \phi^*; \mathcal{P}^*)} f(\hat{\theta}) d\hat{\theta}} & \text{if } \theta \in K' \cap X(\sigma^*, \phi^*; \mathcal{P}^*), \\ 0 & \text{otherwise.} \end{cases} \tag{B.9}$$

By Lemma 1, $\phi'(\sigma'(\theta)) = y_1$ for any $\theta \in \Theta_{11}$, so $K' \cap \Theta_{11} = \emptyset$. If there exists type $\theta' \in K' \cap (\Theta_{21} \cup \Theta_{01})$, then $v(\theta', \phi'(\sigma'(\theta'))) < v(\theta', \phi^*(\sigma^*(\theta)))$ holds, which contradicts Condition (U2). Hence, $K' \cap (\Theta_{21} \cup \Theta_{01}) = \emptyset$. That is, $K' \subseteq \Theta_{22} \cup \Theta_{12} \cup (\Theta_0 \backslash \Theta_{01})$. Notice that:

- if $\theta \in K' \cap X(\sigma^*, \phi^*; \mathcal{P}^*)$, then $v(\theta, \phi'(\sigma'(\theta))) > v(\theta, \phi^*(\sigma^*(\theta)))$, and
- if $\theta \in K' \backslash X(\sigma^*, \phi^*; \mathcal{P}^*)$, then $v(\theta, \phi'(\sigma'(\theta))) = v(\theta, \phi^*(\sigma^*(\theta)))$.

Now, we define function $\pi' : \Theta \to [0, 1]$ by:

$$\pi'(\theta) \equiv \begin{cases} 1 & \text{if } \theta \in K' \cap X(\sigma^*, \phi^*; \mathcal{P}^*), \\ 0 & \text{otherwise.} \end{cases} \tag{B.10}$$

However, for any $\theta \in K'$:

$$\frac{\pi'(\theta)f(\theta)}{\int_\Theta \pi'(\hat{\theta})f(\hat{\theta})d\hat{\theta}} = \begin{cases} \frac{f(\theta)}{\int_{K'\cap X(\sigma^*,\phi^*;\mathcal{P}^*)} f(\hat{\theta})d\hat{\theta}} & \text{if } \theta \in K' \cap X(\sigma^*,\phi^*;\mathcal{P}^*), \\ 0 & \text{otherwise} \end{cases}$$

$$= \mathcal{P}^*(\theta|m'), \tag{B.11}$$

which is a contradiction to that PBE $(\sigma',\phi';\mathcal{P}')$ defeats PBE $(\sigma^*,\phi^*;\mathcal{P}^*)$. ■

B.1.4 Credible message equilibrium

Rabin (1990) suggests a notion of credible message equilibrium that is a PBE constructed by a sense of rationalizable strategies that represent the minimum level of information that the sender can credibly transmit. As a result, only the least informative equilibrium survives, as shown below. Let $\mathcal{X} \equiv \{ X_1, X_2, \cdots, X_D \}$ such that (i) $X_i \subseteq \Theta$ for any i and (ii) $X_i \cap X_j = \emptyset$ for any $i \neq j$ be a *type profile*. Notice that a type profile may not be a partition of the type space. $\Theta_\mathcal{X} \equiv \{ \theta \in \Theta \mid \exists X_i \in \mathcal{X}$ such that $\theta \in X_i \}$ represents the set of types covered by type profile $\mathcal{X}$. $M(X_i)$ represents the set of *self-claiming messages* insisting that "my type is in X_i". Notice that because we consider a persuasion game with fully certifiable information, $M(X_i) = \{X_i\}$ for any $X_i \in \mathcal{X}$. Let $Y^*(X_i) \equiv \left\{ y^* \in Y \,\middle|\, y^* \in \arg\max_{y\in Y} \int_{X_i} u(\theta,y)f(\theta)d\theta \right\}$ be the set of the receiver's best responses given that her prior belief is restricted to subset X_j. A *credible message profile* $\mathcal{X}$ is a type profile such that for any $X_i \in \mathcal{X}$ and $\theta \in X_i$, $Y^*(X_i) = \arg\max_{y\in Y} v(\theta,y)$.[29] In general, there exist multiple credible message profiles, so we focus on the maximal one in the following sense. Let $\mathcal{X}^*$ be a credible message profile such that (i) for any credible message profile $\mathcal{X}$, $\Theta_\mathcal{X} \subseteq \Theta_{\mathcal{X}^*}$, and (ii) for any $X_i, X_j \in \mathcal{X}^*$, $Y^*(X_i) \neq Y^*(X_j)$. We regard credible message profile $\mathcal{X}^*$ as the minimal amount of information that the sender can transmit, and apply the iterative elimination procedure of strictly dominated strategies consistently with it as follows. We say that strategy σ is *strictly dominated in* $\Sigma' \times \Phi'$ where $\Sigma' \subseteq \Sigma$ and $\Phi' \subseteq \Phi$ if there exists strategy $\sigma' \in \Sigma'$ such that for any distribution π over Φ', $\sum_{\phi\in\Phi'} \pi(\phi)\bar{V}(\theta,\phi(\sigma'(\theta))) > \sum_{\phi\in\Phi'} \pi(\phi)\bar{V}(\theta,\phi(\sigma(\theta)))$. The strict dominance for the receiver is defined likewise. Given credible message profile $\mathcal{X}^*$, let Σ^0_M and Φ^0_M

[29] Rabin (1990) requires additional condition that even if types not covered by type profile $\mathcal{X}$ may mimic types in $X_i \in \mathcal{X}$, the set of the receiver's best responses is equivalent to $Y^*(X_i)$. However, this condition is always satisfied in our environment.

be the sets of strategies of the sender and the receiver defined by:

$$\Sigma_M^0 \equiv \left\{ \sigma \in \Delta(M)^\Theta \;\middle|\; \sigma(\theta) = X_i \;\forall X_i \in \mathcal{X}^* \text{ and } \theta \in X_i \right\}, \tag{B.12}$$

$$\Phi_M^0 \equiv \left\{ \phi \in \Delta(Y)^M \;\middle|\; \phi(m) \in \Delta(Y^*(X_i)) \;\forall X_i \in \mathcal{X}^* \text{ and } m = X_i \right\}, \tag{B.13}$$

respectively. For $n \geq 1$, Σ_M^n and Φ_M^n are recursively defined as follows:

$$\Sigma_M^n \equiv \left\{ \sigma \in \Sigma_M^{n-1} \;\middle|\; \sigma \text{ is not strictly dominated in } \Sigma_M^{n-1} \times \Phi_M^{n-1} \right\}, \tag{B.14}$$

$$\Phi_M^n \equiv \left\{ \phi \in \Phi_M^{n-1} \;\middle|\; \phi \text{ is not strictly dominated in } \Sigma_M^{n-1} \times \Phi_M^{n-1} \right\}. \tag{B.15}$$

Define $\Sigma_M^{\mathcal{X}^*} \equiv \Sigma_M^\infty$ and $\Phi_M^{\mathcal{X}^*} \equiv \Phi_M^\infty$.[30]

Definition 9 *Credible Message Equilibrium (Rabin, 1990)*

(i) A set of strategy pairs $\Sigma_M \times \Phi_M$ defined as follows is called credible message rationalizable strategies:

$$\Sigma_M \times \Phi_M \equiv \begin{cases} \Sigma_M^{\mathcal{X}^*} \times \Phi_M^{\mathcal{X}^*} & \textit{if } \exists \textit{ a credible message profile } \mathcal{X}^*; \\ \Sigma_R \times \Phi_R & \textit{otherwise,} \end{cases} \tag{B.16}$$

where $\Sigma_R \times \Phi_R$ is a set of pairs of rationalizable strategies.

(ii) A PBE $(\sigma^*, \phi^*; \mathcal{P}^*)$ is a credible message equilibrium if $(\sigma^*, \phi^*) \in \Sigma_M \times \Phi_M$.

Proposition 10 *U^- is the unique informativeness selected by the credible message equilibrium.*[31]

Proof. Because of Assumptions 1 and 2, the maximal credible message profile $\mathcal{X}^*$ is uniquely determined except for types in Θ_0. For example, consider the following type profile: $\mathcal{X}^* = \{ X_1, X_2 \}$ where $X_1 \equiv \Theta_{11} \cup \Theta_{21} \cup \Theta_{00} \cup \Theta_{01} \cup \Theta_{10}$ and $X_2 \equiv \Theta_{22} \cup \Theta_{12} \cup \Theta_{02} \cup \Theta_{20}$. Hence, it is trivial that:

$$\Sigma_M = \Sigma_M^0 = \left\{ \sigma \in \Delta(M)^\Theta \;\middle|\; \sigma(\theta) = \begin{cases} X_1 & \text{if } \theta \in X_1, \\ X_2 & \text{otherwise,} \end{cases} \right\} \tag{B.17}$$

$$\Phi_M = \Phi_M^0 = \left\{ \phi \in \Delta(Y)^M \;\middle|\; \phi(m) = \begin{cases} y_1 & \text{if } m = X_1, \\ y_2 & \text{if } m = X_2. \end{cases} \right\} \tag{B.18}$$

[30]Because we consider a infinite game, the iterative elimination procedure may not stop within finite time. However, it does not matter to our result as shown in the following proposition.

[31]If Assumption 2-(ii) is violated, then we have to modify Condition (i) of the maximal credible message profile $\mathcal{X}^*$ as follows: (i') there never exists credible message profile $\mathcal{X}'$ such that $\Theta_{\mathcal{X}^*} \subsetneq \Theta_{\mathcal{X}'}$. With this modification, we can obtain the similar uniqueness result.

Therefore, it is also straightforward that a PBE constructed by credible message rationalizable strategies must be the least informative equilibrium. ■

B.1.5 Proposal-proof equilibrium

Proposal-proof equilibrium developed by Zapater (1997) is a selection criterion based on rationalizability like the credible proposal equilibrium. In contrast with Rabin (1990) who specifies the minimum amount of information that the sender can transmit, Zapater (1997) defines *credible proposal* that is a set of strategies representing the maximum amount of information that the sender will transmit, and then checks whether a credible proposal upsets a PBE as a status quo. While credible proposals exclude weakly dominated messages like certifiable dominance, the selection result is completely different from ours; that is, the proposal-proofness has no bite in our environment as shown following. Again, we focus on pure strategy equilibria in this section following Zapater (1997).

$\bar{\Sigma}$ and $\bar{\Phi}$ represent the sets of sender's and the receiver's pure strategies, respectively. Given $\Sigma' \subseteq \bar{\Sigma}$ and $\phi \in \Delta(\bar{\Phi})$, we say that σ *is a best response to* ϕ *in* Σ' if $\sigma(\theta) \in \arg\max_{m \in \Sigma'(\theta)} \bar{V}(\theta, \phi(m))$ for any $\theta \in \Theta$, where $\Sigma'(\theta) \equiv \bigcup_{\sigma' \in \Sigma'} \{\sigma'(\theta)\}$. Likewise, given $\Phi' \subseteq \bar{\Phi}$ and $m \in \bigcup_{\theta \in \Theta} S(\sigma(\theta))$ for some $\sigma \in \Delta(\bar{\Sigma})$, we say that $\phi(m)$ *is a best response to* m *in* Φ' if $\phi(m) \in \arg\max_{y \in \Phi'(m)} \mathbb{E}_{\mathcal{P}(m,\sigma)}[u(\theta, y)]$, where $\Phi'(m) \equiv \bigcup_{\phi \in \Phi'} \{\phi(m)\}$ and $\mathcal{P}(m, \sigma)$ is the receiver's belief upon observing message m derived from σ consistently with Bayes' rule. We call $\mathrm{P} = (\Sigma_P, \Phi_P) \subseteq (\bar{\Sigma}, \bar{\Phi})$ a *proposal* if it satisfies the following conditions: (i) $\sigma \in \Sigma_P$ if and only if there exists a strategy $\phi \in \Delta(\Phi_P)$ such that σ is a best response to ϕ in $\bar{\Sigma}$; and (ii) $\phi \in \Phi_P$ if and only if there exists a strategy $\sigma \in \Delta(\Sigma_P)$ such that for any $m \in \bigcup_{\theta \in \Theta} \{\sigma(\theta)\}$, $\phi(m)$ is a best response to m in Φ_P. In general, there exist multiple proposals, so we focus on proposals excluding weakly dominated messages. Given two proposals $P = (\Sigma_P, \Phi_P)$ and $Q = (\Sigma_Q, \Phi_Q)$ and messages $m \in \Sigma_P(\theta)$ and $m' \in \Sigma_Q(\theta)$, we denote $m \succsim_\theta m'$ if it satisfies the following condition: for any $\phi \in \Phi_P$ and $\phi' \in \Phi_Q$, $v(\theta, \phi(m)) \geq v(\theta, \phi'(m'))$ with strictly inequality for some $\hat{\phi} \in \Phi_P$ and $\hat{\phi}' \in \Phi_Q$. We say that proposal $P = (\Sigma_P, \Phi_P)$ has the *strong best response property* if for any $\theta \in \Theta$, there exist no messages $m, m' \in \Sigma_P(\theta)$ such that $m \succsim_\theta m'$, and let $\mathbb{P}$ be the set of proposals having the strong best response property.

We consider the following iterative elimination procedure. As an initial point, let Σ_K^0 and Φ_K^0 be the sets of the sender's and the receiver's strategies, respectively, defined by: (i) $\Sigma_K^0 \equiv \Lambda_K$ where Λ_K is the set of strategies such that behaviors of types in proper subset $K \subsetneq \Theta$ are restricted, but types in $\Theta \backslash K$ can send any available messages; and (ii) $\Phi_K^0 \equiv \bar{\Phi}$ with $\Phi_K^0(m) \equiv Y$ for any $m \in M$.

For $n \geq 1$, Σ_K^n and Φ_K^n are recursively defined as follows:

$$\Sigma_K^n \equiv \left\{ \sigma \in \Sigma_K^{n-1} \;\middle|\; \exists \phi \in \Delta(\Phi_K^{n-1}) \text{ such that } \sigma \text{ is a best response to } \phi \text{ in } \Sigma_K^{n-1} \right\}; \tag{B.19}$$

if message m is such that $m \notin \bigcup_{\sigma \in \Sigma_K^{n-1}} \bigcup_{\theta \in \Theta} \{\sigma(\theta)\}$, then $\Phi_K^n(m) \equiv \Phi_K^{n-1}(m)$; otherwise:

$$\Phi_K^n(m) \equiv \left\{ y \in \Phi_K^{n-1}(m) \;\middle|\; \begin{array}{l} y \text{ is a best response to } m \text{ in } \Phi_K^{n-1} \text{ for some } \sigma \in \Delta(\Sigma_K^{n-1}) \\ \text{such that } m \in \displaystyle\bigcup_{\theta \in \Theta} S(\sigma(\theta)) \end{array} \right\}; \tag{B.20}$$

$$\Phi_K^n \equiv \left\{ \phi \in \Phi_K^{n-1} \;\middle|\; \begin{array}{l} \exists\, \sigma \in \Delta(\Sigma_K^{n-1}) \text{ such that} \\ \text{(i) if } m \in \displaystyle\bigcup_{\theta \in \Theta} S(\sigma(\theta)), \text{ then } \phi(m) \text{ is a best response to } m \text{ in } \Phi_K^{n-1}; \\ \text{(ii) otherwise, } \phi(m) \in \Phi_K^n(m). \end{array} \right\} \tag{B.21}$$

Define $\Sigma_K^* \equiv \Sigma_K^\infty$ and $\Phi_K^* \equiv \Phi_K^\infty$. We say that pair (K, Λ_K) *generates* proposal $P = (\sigma_P, \Phi_P) \in \mathbb{P}$ either one of the following conditions holds: (i) if $K \subsetneq \Theta$ and $(\Sigma_K^*, \Phi_K^*) = (\Sigma_P, \Phi_P)$; or (ii) $K = \Theta$ and $\Lambda_K = \Sigma_P$.

Based on the iterative elimination procedure defined above, we further eliminate weakly dominated proposals from $\mathbb{P}$ as follows. For two proposals $P, Q \in \mathbb{P}$ and $K \subseteq \Theta$, we denote $P \succsim_K Q$ if there exist strategies $\sigma \in \Sigma_P$ and $\sigma' \in \Sigma_Q$ such that $\sigma(\theta) \succsim_\theta \sigma'(\theta)$ for any $\theta \in K$. Likewise, we denote $P \sim_K Q$ if the following conditions are satisfies: (i) $\Sigma_P|_K = \Sigma_Q|_K$ where $\Sigma'|_K$ represents the projection of Σ' on subset K; and (ii) for any message $m \in \bigcup_{\theta \in \Theta} (\Sigma_P(\theta) \cup \Sigma_Q(\theta))$, either (a) $\Pi_P(m) = \Pi_Q(m)$; or (b) $\phi(m) = \phi'(m)$ for any $\phi \in \Phi_P$ and $\phi' \in \Phi_Q$ where:

$$\Pi_P(m) \equiv \left\{ \mathcal{P}(m) \in \Delta(\Theta) \;\middle|\; \begin{array}{l} \exists\, \sigma \in \Delta(\Sigma_P) \text{ such that (i) } m \in \bigcup_{\theta \in \Theta} S(\sigma(\theta)), \text{ and} \\ \mathcal{P}(m) \text{ is derived from } \sigma \text{ consistently with Bayes' rule} \end{array} \right\}. \tag{B.22}$$

We say that proposal $Q \in \mathbb{P}$ is *dominated* if there exists pair (K, Λ_K) that generates proposal $P \in \mathbb{P}$ such that (i) for any $\theta \in K$, either $P \succsim_\theta Q$ or $P \sim_\theta Q$; and (ii) for some $\theta' \in K$, $P \succsim_{\theta'} Q$. Let $\mathbb{P}^U$ denote the set of undominated proposals satisfying the strong best response property.

Definition 10 *Proposal-Proof Equilibrium (Zapater, 1997)*

(i) Proposal $P \in \mathbb{P}^U$ is credible if for any $Q \in \mathbb{P}^U$, $\Sigma_Q \subseteq \Sigma_P$. Let $\mathbb{P}^*$ be the set of credible

proposals.

(ii) *A PBE $e^* \equiv (\sigma^*, \phi^*; \mathcal{P}^*)$ is proposal proof if either one of the following conditions hold:*

(P1) *there exists no pair (K, Λ_K) that generates a credible proposal $P \in \mathbb{P}^*$ such that $P \succsim_K e^*$; or*

(P2) *e^* is a PBE of a game where the players' strategies are constrained to Σ_P and Φ_P for some credible proposal $P \in \mathbb{P}^*$.*

Proposition 11 *For any $U \in [U^-, U^+]$, there exists a proposal-proof equilibrium $e^* = (\sigma^*, \phi^*; \mathcal{P}^*)$ whose informativeness is U.*

Proof. Suppose, in contrast, that there exists $U \in [U^-, U^+]$ such that any PBE whose informativeness is U is not proposal proof. Fix PBE $e^* = (\sigma^*, \phi^*; \mathcal{P}^*)$ whose informativeness is U, arbitrarily. Because e^* is not proposal proof, there must exists pair (K, Λ_K) generating a credible proposal P such that $P \succsim_K e^*$. If $P(X(e^*)) = 0$, then there exists another PBE e' such that $\mathbb{E}[u(\theta, \phi'(\sigma'(\theta)))] = U$ and $\phi'(\sigma'(\theta)) = y^S(\theta)$ for any $\theta \in \Theta$. However, because $P \not\succsim_\theta e'$ for any credible proposal P and $\theta \in \Theta$, e' is proposal proof, which is a contradiction. Hence, $P(X(e^*)) > 0$ must hold. By Lemma 5, we can restrict our attention to $X(e^*) \subseteq \Theta_{12}$ without loss of generality. Also, by Lemma 1, $v(\theta, \phi^*(\sigma^*(\theta))) = v(\theta, y^S(\theta))$ for any $\theta \in \Theta_{11} \cup \Theta_{22}$. Hence, to hold $P \succsim_K e^*$, $K \subseteq \Theta_{12}$ must hold. Now, we consider the following strategies. Define subset $A \subseteq \Theta_{11} \cup \Theta_{21}$ such that (i) $\mathbb{E}[u(\theta, y_1)|A] \geq \mathbb{E}[u(\theta, y_2)|A]$; and (ii) $P(A \cap \Theta_{11}) > 0$ and $P(A \cap \Theta_{21}) > 0$. Let Σ_1, Σ_2, Φ_1, and Φ_2 are the sets of strategies defined by:

$$\Sigma_1 \equiv \left\{ \sigma \in \bar{\Sigma} \;\middle|\; \sigma(\theta) = A \text{ for any } \theta \in A \right\}, \tag{B.23}$$

$$\Sigma_2 \equiv \left\{ \sigma \in \bar{\Sigma} \;\middle|\; \sigma(\theta) = \left\{ \begin{array}{cl} A \cap \Theta_{11} & \text{if } \theta \in A \cap \Theta_{11}, \\ A & \text{if } \theta \in A \cap \Theta_{21} \end{array} \right. \right\} \tag{B.24}$$

$$\Phi_1 \equiv \left\{ \phi \in \bar{\Phi} \;\middle|\; \phi(A) = y_1 \right\}, \tag{B.25}$$

$$\Phi_2 \equiv \left\{ \phi \in \bar{\Phi} \;\middle|\; \phi(A) = y_2 \right\}. \tag{B.26}$$

Lemma 6 *For any proposal P generated by (K, Λ_K), $\Sigma_i \cap \Sigma_P \neq \emptyset$ and $\Phi_j \cap \Phi_P \neq \emptyset$ for any $i, j \in \{1, 2\}$.*

Proof of Lemma 6. It is sufficient to show that for any $n \in \mathbb{N}$, $\Sigma_i \cap \Sigma_K^n \neq \emptyset$ and $\Phi_j \cap \Phi_K^n \neq \emptyset$ for any $i, j \in \{1, 2\}$. We show the statement by induction.

(i) $n = 0$.

Because $K \subseteq \Theta_{12}$, behaviors of types in $\Theta_{11} \cup \Theta_{21}$ are not restricted in the initial point. Hence, it is obvious that $\Sigma_i \cap \Sigma_K^0 \neq \emptyset$ for any i. Also, because $\Phi_K^0 = \bar{\Phi}$, $\Phi_j \cap \Phi_K^0 \neq \emptyset$ trivially holds for any j.

(ii) $n = t + 1$.

By the inductive hypothesis, we assume that $\Sigma_i \cap \Sigma_K^t \neq \emptyset$ and $\Phi_j \cap \Phi_K^t \neq \emptyset$ for any i, j. Notice that for any $\theta \in A$, sending message $m = A$ is a best response to strategy $\phi_1 \in \Phi_1 \cap \Phi_K^t$. Hence, $\Sigma_1 \cap \Sigma_K^{t+1} \neq \emptyset$ holds. Also, because $\Sigma_2 \cap \Sigma_K^t \neq \emptyset$, $\Phi_K^{t+1}(A \cap \Theta_{11}) = \{y_1\}$. That is, there exists strategy $\phi' \in \Phi_1 \cap \Phi_K^t$ such that $\phi'(A) = \phi'(A \cap \Theta_{11}) = y_1$. Therefore, we can conclude that $\Sigma_2 \cap \Sigma_K^{t+1} \neq \emptyset$. By the induction hypothesis, there exist strategies $\sigma_1 \in \Sigma_1 \cap \Sigma_K^t$ and $\sigma_2 \in \Sigma_2 \cap \Sigma_K^t$. Given strategy σ_1, the receiver's best response to message $m = A$ is choosing action $y = y_1$. That is, $\Phi_1 \cap \Phi_K^{t+1} \neq \emptyset$ holds. Likewise, given strategy σ_2, the receiver's best response to message $m = A$ is choosing action $y = y_2$. Thus, $\Phi_2 \cap \Phi_k^{t+1} \neq \emptyset$. $\square$

By Lemma 6, there exist strategies $\sigma', \sigma'' \in \Sigma_P$ and $\phi' \in \Phi_P$ such that $\sigma'(\theta) = A$ for any $\theta \in A \cap \Theta_{11}$; $\sigma''(\theta) = A \cap \Theta_{11}$ for any $\theta \in A \cap \Theta_{11}$; and $\phi'(A) = y_2$. However, because $\phi(A \cap \Theta_{11}) = y_1$ for any $\phi \in \Phi_P$, for any $\theta \in A \cap \Theta_{11}$, (i) $v(\theta, \phi(A \cap \Theta_{11})) \geq v(\theta, \hat{\phi}(A))$ for any $\phi, \hat{\phi} \in \Phi_P$; and (ii) $v(\theta, \phi'(A \cap \Theta_{11})) > v(\theta, \phi'(A))$ hold. That is, $A \cap \Theta_{11} \succsim_\theta A$ where $A, A \cap \Theta_{11} \in \Sigma_P(\theta)$ for any $\theta \in A \cap \Theta_{11}$. This implies that credible proposal P does not have the strong best response property, which is a contradiction. ■

B.2 Δ-rationalizability when either $\Theta_{12} = \emptyset$ or $\Theta_{21} = \emptyset$

In this subsection, we show that restrictions to the receiver's skeptical beliefs guarantee the unique selection when either $\Theta_{12} = \emptyset$ or $\Theta_{21} = \emptyset$. Throughout this subsection, we assume that $\Theta_{12} \neq \emptyset$ and $\Theta_{21} = \emptyset$. Define:

$$\Delta^S \equiv \Delta\left(\bar{\Phi}\right), \tag{B.27}$$

$$\Delta^R \equiv \left\{ \mu^R \in \bar{\Delta}(\Theta, M) \;\middle|\; \forall m \in M \text{ such that } m \cap \Theta_{12} \neq \emptyset,\ \mu^R(\Theta_{12}|m) = 1 \right\}, \tag{B.28}$$

$$\Phi_S \equiv \left\{ \phi \in \Phi_\Delta^{-1} \;\middle|\; \begin{array}{l} \text{(i) if } m \cap \Theta_{12} \neq \emptyset, \text{ then } \phi(m) = y_1, \\ \text{(ii) otherwise, } \phi(m) \in \bigcup_{\theta \in m} \{y^R(\theta)\} \end{array} \right\}, \tag{B.29}$$

$$G_S \equiv \left\{ (\theta, m) \in \Sigma_\Delta^0 \;\middle|\; \text{if } \theta \in \Theta_{22}, \text{ then } m \cap \Theta_{12} = \emptyset \right\}. \tag{B.30}$$

Proposition 12 $\Phi_\Delta^1 = \Phi_S$.

Proof. For message $m \in M$ such that $m \cap \Theta_{12} = \emptyset$, we can show the statement using the similar argument as in Proposition 5. Hence, we can restriction attention to m such that $m \cap \Theta_{12} \neq \emptyset$. First, we show that $\Phi^1_\Delta \subseteq \Phi_S$. Suppose, in contrast, that there exists $\phi \in \Phi^1_\Delta$ such that $\phi \notin \Phi_S$. That is, there exists $m \in M$ such that $m \cap \Theta_{12} \neq \emptyset$ and $\phi(m) = y_2$. However, for any $\mu^R \in \Delta^R$ such that $\mu^R(\Theta^1_\Delta(m)|m) = 1$, $BR_R(m, \mu^R(m)) = \{y_1\}$ holds, which contradicts $\phi \in \Phi^1_\Delta$. The converse can be shown by the similar argument used above. Therefore, $\Phi^1_\Delta = \Phi_S$. ■

Proposition 13 $\Sigma^2_\Delta = G_S$.

Proof. We can show that $\Sigma^2_\Delta \subseteq G_S$ using the similar argument as in the proof of Proposition 6. Hence, we only show the converse. Fix $(\theta, m) \in G_S$, arbitrarily. First, suppose that $\theta \in \Theta_{22}$, and then $m \cap \Theta_{12} = \emptyset$. Because for any $m \in M(\theta)$ such that $m \cap \Theta_{12} = \emptyset$, there exists $\phi_m \in \Phi^1_\Delta$ such that $\phi_m(m) = y_2$. Hence, for belief $\mu^S_m \in \Delta^S$ such that $\mu^S_m(\phi_m) = 1$, $\sum_{y \in Y} v(\theta, y)\pi(y|m, \mu^S_m) = v(\theta, y^S(\theta))$. Thus, $m \in BR_S(\theta, \mu^S_m)$ holds. Second, suppose that $\theta \in \Theta_{12}$. Notice that $m \cap \Theta_{12} \neq \emptyset$ holds for any $m \in M(\theta)$. By Proposition 12, $\phi(m) = y_1$ for any $\phi \in \Phi^1_\Delta$ and $m \in M(\theta)$. That is, it is obvious that $m \in BR_S(\theta, \mu^S)$ holds for any $m \in M(\theta)$ and $\mu^S \in \Delta\left(\Phi^1_\Delta\right) = 1$. Third, suppose that $\theta \in \Theta_{11} \cup \Theta_{01}$. By Proposition 12, for any $m \in M(\theta)$, there exists $\phi_m \in \Phi^1_\Delta$ such that $\phi_m(m) = y_1$. Hence, for belief $\mu^S_m \in \Delta^S$ such that $\mu^S(\phi_m) = 1$, $\sum_{y \in Y} v(\theta, y)\pi(y|m, \mu^S_m) = v(\theta, y^S(\theta))$. Thus, $m \in BR_S(\theta, \mu^S_m)$ holds. Forth, suppose that $\theta \in \Theta_{02}$. By Proposition 12, there exists $\phi' \in \Phi^1_\Delta$ such that $\phi'(m) = y_1$ for any $m \in M(\theta)$. Hence, under belief $\mu^{S'} \in \Delta^S$ such that $\mu^{S'}(\phi') = 1$, $m \in BR_S(\theta, \mu^{S'})$ holds for any $m \in M(\theta)$. Finally, suppose that $\theta \in \Theta_0 \backslash (\Theta_{01} \cup \Theta_{02})$. In this scenario, both actions y_1 and y_2 are indifferent for type θ. Thus, it is obvious that $m \in BR_S(\theta, \mu^S)$ for any $m \in M(\theta)$ and $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi^1_\Delta\right) = 1$. Therefore, $(\theta, m) \in \Sigma^2_\Delta$ holds; that is, $\Sigma^2_\Delta = G_S$. ■

B.2.1 Proof of Theorem 3

The uniqueness can be shown by the similar arguments used in the body of the paper. Hence, it remains to show the existence. Define:

$$\sigma^*(\theta) \equiv \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0, \\ \Theta_{12} & \text{if } \theta \in \Theta_{12}, \end{cases} \tag{B.31}$$

$$\Phi^* \equiv \left\{ \phi \in \bar{\Phi} \;\middle|\; \phi(m) = \begin{cases} y^R(\theta) & \text{if } m = \{\theta\}, \\ y_1 & \text{if } m \cap \Theta_{12} \neq \emptyset. \end{cases} \right\}. \tag{B.32}$$

Because for any $\phi^* \in \Phi^*$, (σ^*, ϕ^*) constructs a PBE whose informativeness is U^+. It is then sufficient to show that $G(\sigma^*) \subseteq \Sigma_\Delta^{2k}$ and $\Phi^* \cap \Phi_\Delta^{2k-1} \neq \emptyset$ for any $k \geq 0$. We show the statement by induction. If $k = 0$, then the statement is trivial because $\Sigma_\Delta^0 = \bigcup_{\sigma \in \bar{\Sigma}} G(\sigma)$ and $\Phi_\Delta^{-1} = \bar{\Phi}$. We then assume that $G(\sigma^*) \subseteq \Sigma_\Delta^{2k}$ and $\Phi^* \cap \Phi_\Delta^{2k-1} \neq \emptyset$ for some $k > 0$ as an induction hypothesis.

First, we show that $\Phi^* \cap \Phi_\Delta^{2k+1} \neq \emptyset$. Suppose, in contrast, that $\Phi^* \cap \Phi_\Delta^{2k+1} = \emptyset$. That is, for any $\phi \in \Phi_\Delta^{2k+1}$, either one of the following holds: (i) there exists message $m' = \{\theta\}$ such that $\phi(m') \neq y^R(\theta)$, or (ii) there exists message m'' such that $m'' \cap \Theta_{12} \neq \emptyset$ and $\phi(m'') = y_2$. Fix $\phi \in \Phi_\Delta^{2k+1}$, arbitrarily. If the first scenario holds, then $\phi \notin \Phi_\Delta^{2k+1}$ holds because of Proposition 12 and $\Phi_\Delta^{2k+1} \subseteq \Phi_\Delta^1$, which is a contradiction. We then suppose that the second scenario holds. Notice that as long as the receiver's beliefs are restricted to Δ^R, for any type $\theta \in \Theta_{12}$, any available message can be a best response in any round of the elimination process. Hence, $(m'' \cap \Theta_{12}) \subseteq \Theta_\Delta^{2k+1}(m'')$ holds. Define:

$$M'' \equiv \left\{ m'' \in M \;\middle|\; \text{(i) } m'' \cap \Theta_{12} \neq \emptyset, \text{ and (ii) } \phi(m'') = y_2 \right\}. \tag{B.33}$$

For any $m'' \in M''$, there exists $\theta'' \in \Theta_\Delta^{2k+1}(m'') \cap \Theta_{12}$. Hence, for belief $\tilde{\mu}^R \in \Delta^R$ such that $\tilde{\mu}^R(\theta''|m'') = 1$ for any $m'' \in M''$, $y_1 \in BR_R(m'', \tilde{\mu}^R(m''))$ holds. Then, define:

$$\tilde{\phi}(m) \equiv \begin{cases} y_1 & \text{if } m \in M'', \\ \phi(m) & \text{otherwise.} \end{cases} \tag{B.34}$$

By construction, $\tilde{\phi}$ is not eliminated in the previous rounds, and then $\tilde{\phi} \in \Phi^* \cap \Phi_\Delta^{2k+1}$ holds, which is a contradiction. Therefore, $\Phi^* \cap \Phi_\Delta^{2k+1} \neq \emptyset$ holds.

Next, we show that $G(\sigma^*) \subseteq \Sigma_\Delta^{2k+2}$. First, suppose that $\theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0$. Because $\Phi^{2k+1} \subseteq \Phi_\Delta^1$, $\phi(\{\theta\}) = y^R(\theta)$ holds for any $\phi \in \Phi_\Delta^{2k+1}$. Hence, there exists belief $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi_\Delta^{2k+1}\right) = 1$ and $\sum_{y \in Y} v(\theta, y)\pi(y|\{\theta\}, \mu^S) = v(\theta, y^S(\theta))$; that is, $\{\theta\} \in BR_S(\theta, \mu^S)$ holds. We then suppose that $\theta \in \Theta_{12}$. Again, because type θ cannot induce action y_2 with positive probability, it is obvious that $\Theta_{12} \in BR_S(\theta, \mu^S)$ holds for any $\mu^S \in \Delta^S$ such that $\mu^S\left(\Phi_\Delta^{2k+1}\right) = 1$. Thus, we can conclude that $G(\sigma^*) \subseteq \Sigma_\Delta^{2k+2}$. ■

B.3 Proof of Theorem 4

B.3.1 Preliminaries

Proposition 14

(i) In Case 1:

$$\Sigma_C = \hat{\Sigma}_1 \equiv \left\{ \sigma \in \Sigma \;\middle|\; \begin{array}{l} \forall m \in S(\sigma(\theta)), \\ m \begin{cases} > \theta^R & \textit{if } \theta \in \Theta_{22}, \\ \leq \theta^R & \textit{if } \theta \in \Theta_{21} \end{cases} \end{array} \right\}. \tag{B.35}$$

(ii) In Case 2:

$$\Sigma_C = \hat{\Sigma}_2 \equiv \left\{ \sigma \in \Sigma \;\middle|\; \forall \theta \in \Theta_{22} \textit{ and } m \in S(\sigma(\theta)),\ m > \theta^R \right\}. \tag{B.36}$$

Proof. (i) ($\Sigma_C \subseteq \hat{\Sigma}_1$) Suppose, in contrast, that there exists $\sigma \in \Sigma_C$ such that $\sigma \notin \hat{\Sigma}_1$.

Case (a): there exist $\theta' \in \Theta_{22}$ and $m' \in S(\sigma(\theta'))$ such that $m \leq \theta^R$.

Consider the following strategy σ' defined by:

$$\sigma'(\theta) \equiv \begin{cases} \theta' & \text{if } \theta = \theta', \\ \sigma(\theta) & \text{otherwise} \end{cases} \tag{B.37}$$

Notice that there exists $\phi' \in \Phi_C$ such that $\phi'(m') = y_1$, and $\phi(\theta') = y_2$ holds for any $\phi \in \Phi_C$. However, it implies that $\bar{V}(\theta, \phi(\sigma'(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ holds for any $\theta \in \Theta$ and $\phi \in \Phi_C$, and the strict inequality holds for $\theta' \in \Theta$ and $\phi' \in \Phi_C$, which contradicts $\sigma \in \Sigma_C$.

Case (b): there exist $\theta'' \in \Theta_{21}$ and $m'' \in S(\sigma(\theta''))$ such that $m'' > \theta^R$.

Consider the following strategies σ'' defined by:

$$\sigma''(\theta) = \begin{cases} \theta^R & \text{if } \theta = \theta'', \\ \sigma(\theta) & \text{otherwise} \end{cases} \tag{B.38}$$

Notice that there exists $\phi'' \in \Phi_C$ such that $\phi''(\theta^R) = y_1$, and $\phi(m'') = y_2$ holds for any $\phi \in \Phi_C$. However, it implies that $\bar{V}(\theta, \phi(\sigma''(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ holds for any $\theta \in \Theta$ and $\phi \in \Phi_C$, and the strict inequality holds for $\theta'' \in \Theta$ and $\phi'' \in \Phi_C$, which contradicts $\sigma \in \hat{\Sigma}_C$.

($\Sigma_C \supseteq \hat{\Sigma}_1$) Suppose, in contrast, that there exists $\sigma \in \hat{\Sigma}_1$ such that $\sigma \notin \Sigma_C$. That is, there exists $\sigma' \in \Sigma$ such that (i) $\bar{V}(\theta, \phi(\sigma'(\theta))) \geq \bar{V}(\theta, \phi(\sigma(\theta)))$ holds for any $\theta \in \Theta$ and $\phi \in \Phi_C$, and (ii) $\bar{V}(\theta', \phi'(\sigma'(\theta'))) > \bar{V}(\theta', \phi'(\sigma(\theta')))$ holds for some $\theta' \in \Theta$ and $\phi' \in \Phi_C$. Because of Condition (ii), $\theta' \notin \Theta_{20}$. There are the following cases to be checked.

Case (a): $\theta' \in \Theta_{22}$.

Because $\sigma \in \hat{\Sigma}_1$, $m' > \theta^R$ holds for any $m' \in S(\sigma(\theta))$. However, it implies that $\phi(m') = y_2$ holds for any $\phi \in \Phi_C$ and $m' \in S(\sigma(\theta'))$, which contradicts Condition (ii).

Case (b): $\theta' \in \Theta_{21} \cup \Theta_{11} \cup \Theta_{01}$.

Because $\sigma \in \hat{\Sigma}_1$ and $\sigma' \neq \sigma$, there exists message $m' \in S(\sigma(\theta'))$ such that $m' \leq \theta^R$ and $\sigma(m'|\theta') > \sigma'(m'|\theta')$. Notice that there exists $\phi'' \in \Phi_C$ such that $\phi''(m) = y_2$ for any $m \in S(\sigma'(\theta'))\backslash\{m'\}$, and $\phi''(m') = y_1$. However, it implies that $\bar{V}(\theta', \phi''(\sigma(\theta'))) > \bar{V}(\theta', \phi''(\sigma'(\theta')))$, which contradicts Condition (i).

Therefore, we can conclude that $\Sigma_C = \hat{\Sigma}_1$.

(ii) By the similar argument used in (i), we can show that $\Sigma_C = \hat{\Sigma}_2$. ■

B.3.2 Proof of Theorem 4

Hereafter, we focus on Case 1 because Case 2 can be shown using the similar argument adopted here.[32]

(Existence) It is straightforward that the two-partition equilibrium with cutoff θ^S is a CUE. Hence, it remains to show that it is the most informative equilibrium. Suppose, in contrast, that there exists PBE $(\sigma', \phi'; \mathcal{P}')$ such that $\mathbb{E}[\bar{U}(\theta, \phi'(\sigma'(\theta)))] \equiv U' > U^S$ where U^S is the informativeness of the two-partition equilibrium with cutoff θ^S. Notice that $\phi'(m) = y_2$ should hold for any $\theta \in \Theta_{22}$ and $m \in S(\sigma'(\theta))$; otherwise, such a type deviates to send message $m = \theta$, which induces action y_2 for certain. Define $\Theta'_{21} \equiv \{\, \theta \in \Theta_{21} \mid \exists m \in S(\sigma'(\theta)) \text{ such that } \phi'(y_2|m) > 0 \,\}$. Because $U' > U^S$, $P(\Theta'_{21}) > 0$ must hold. Furthermore, to hold this equilibrium, $\phi'(y_2|m) > 0$ must hold for any $m \leq \theta^R$; otherwise, type $\theta \in \Theta'_{21}$ has an incentive to send a message m' such that $\phi(y_2|m') = 0$. Define $M^S \equiv \{\, m \in M \mid \exists \theta \in \Theta_{11} \cup \Theta_{21} \cup \Theta_{01} \text{ such that } m \in S(\sigma'(\theta)) \,\}$, and assume that it is countable without loss of generality. Because of the properties of PBE $(\sigma', \phi'; \mathcal{P}')$ mentioned above, for any $m \in M^S$:

$$\mathbb{E}_{\mathcal{P}'(m)}[u(\theta, y_1)]\Pr(m|\Theta_{11} \cup \Theta_{21}) \leq \mathbb{E}_{\mathcal{P}'(m)}[u(\theta, y_2)]\Pr(m|\Theta_{11} \cup \Theta_{21}). \tag{B.39}$$

However, by summing up both sides of (B.39) for $m \in M^S$, we obtain that $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$, which contradicts Assumption 4-(ii). Therefore, we can conclude that the two-partition equilibrium with cutoff θ^S attains the maximum informativeness.

[32] The details are available from the author upon request.

(Uniqueness) Suppose, in contrast, that there exists CUE $(\sigma', \phi'; \mathcal{P}')$ such that $\mathbb{E}[\bar{U}(\theta, \phi'(\sigma'(\theta)))] \equiv U' \neq U^*$. By Proposition 14, any type in agreement region Θ_{22} is never pooling with types in $\Theta_{11} \cup \Theta_{21} \cup \Theta_0$. Define $\Theta' \equiv \{\, \theta \in \Theta \backslash \Theta_{22} \mid \exists m \in S(\sigma'(\theta))$ such that $\phi'(y_2|m) > 0 \,\}$. Because $U' \neq U^*$, $P(\Theta') > 0$ should hold. Now, suppose, in contrast, that $\Theta' \subseteq \Theta_{11} \cup \Theta_{01}$. By construction, almost every type in $\Theta_{11} \cap \Theta'$ must be pooling with types in Θ_{21}; otherwise, such a type cannot induce action y_2 with positive probability. However, it implies that $\Theta'_{21} \equiv \Theta_{21} \cap \Theta' \neq \emptyset$, which is a contradiction. Hence, $\Theta'_{21} \neq \emptyset$, and then it is necessary for holding this equilibrium that $\phi'(y_2|m) > 0$ for any $m \leq \theta^R$; otherwise, type $\theta \in \Theta'_{21}$ deviates to message m such that $\phi'(y_2|m) = 0$. Then, by the same argument used in the existence part, we can derive a contradiction. Therefore, we conclude that U^* is the unique informativeness supported by CUEs. ■