A Characterization of Equilibrium Set of Persuasion Games with Binary Actions *

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February 7, 2014

Abstract

This paper considers a persuasion game between one sender and one receiver. The perfectly informed sender can fully certify any private information that is drawn from a continuum set, and the receiver has binary actions. We focus on the situation where both full information disclosure and full information suppression are impossible. We characterize the set of pure strategy equilibria in terms of informativeness measured by the receiver's ex ante expected utility in this environment; there exist continuum equilibria. The set is characterized by the most and the least informative equilibria, and then any value between the bounds can be supported in equilibrium with transparent construction of the associated equilibrium.

Journal of Economic Literature Classification Numbers: C72, D82.

Key Words: persuasion game; fully certifiable state; binary actions of the receiver; no full disclosure equilibrium; set of equilibria

*This paper is based on the third chapter of my Ph.D. dissertation at Washington University in St. Louis. I am particularly indebted to John Nachbar, Haluk Ergin and David Levine for their continuous support and encouragement. I am grateful to Rohan Dutta, Philip Dybvig, Hassan Faghani Dermi, Taro Kumano, Wen-Chieh Lee, John Patty, Tsz-Nga Wong, Haibo Xu, Takuro Yamashita, and all participants in the Theory Bag Lunch Seminar. I also appreciate the editor and anonymous referees for valuable comments. All remaining errors are my own.

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1 Introduction

Persuasion games are costless sender–receiver games with certifiable private information. The sender sends a message about his private information to the receiver who chooses an action that affects the players' payoffs.[1] The sender can send any message costlessly, but he cannot misreport the information because the information is certifiable and the sender is required to submit evidence with his message.[2] Hence, the sender manipulates the information by concealing unfavorable information instead of lying.

The aim of this paper is making clear how much information is transmitted in equilibrium when full information disclosure and full information suppression are impossible. The existing literature of persuasion games mainly focuses on the situation where full information disclosure or full information suppression occurs, and discusses the possibilities of these extreme scenarios. However, in our best knowledge, we know little about what happens if these extreme scenarios do not occur. That is, we do not know what are the second-best and second-worst equilibria in terms of information transmission. This is an important issue; for example, without knowing what happens in the second-best or the second-worst scenario, we cannot compare different environments where the extreme scenarios do not occur. This paper then tries to fill the gap by characterizing non-extreme equilibria in terms of transmitted information.

In order to manage the objective, this paper considers a simple persuasion game between one sender and one receiver. The sender is perfectly informed about the state of nature that is drawn from a continuum set, and he can fully certify this information. An important restriction of the model is that the receiver's action is binary; that is, $y = y_1$ or $y = y_2$. Hence, depending on the players' ex post preferences over the actions, the state space is partitioned into the following five regions: (i) both prefers action y_1, (ii) both prefers action y_2, (iii) the sender prefers action y_1, but the receiver prefers action y_2, (iv) the sender prefers action y_2, but the receiver prefers action y_1, and (v) either one of the player is indifferent. We assume that the conflict between the players is nontrivial in the sense that regions (i) to (iv) occur with positive probabilities. Under this assumption, both full information disclosure and full information suppression are never supported in equilibrium.

The main result of this paper is a characterization of the set of pure strategy equilibria from the viewpoint of the informativeness of equilibria measured by the receiver's ex ante expected utility.[3]

[1] As a matter of convention, we treat the sender as male and the receiver as female throughout this paper.

[2] In the literature, such information is called *hard information*.

[3] The definition of the informativeness in this paper is different from that used in the information theory. We follow the definition frequently used in the cheap talk games à la Crawford and Sobel (1982).

First, we specify the most and the least informative equilibria. In the most informative equilibrium, types in agreement regions (i) and (ii) fully disclose, but all types in disagreement regions (iii) and (iv) are pooling. On the other hand, in the least informative equilibrium, types in the disagreement region (iii) (resp. disagreement region (iv)) are pooling with types in agreement region (i) (resp. agreement region (ii)) up to the point where the actions are indifferent for the receiver given the pooling types, and the remaining types disclose. An implication from this characterization is that in any equilibrium, the sender can suppress a part of unfavorable information, but cannot suppress all of them if the players' conflict is nontrivial in the above sense.

Furthermore, we show that any value between the bounds can be supported in equilibrium; that is, there exist continuum equilibria in this setup. Each equilibrium is characterized by a set of types where the information is disclosed to the receiver. Intuitively, this disclosure set is "minimized" in the least informative equilibrium and "maximized" in the most informative equilibrium. We then continuously expand the "minimized" disclosure set until converging to the "maximized" one. For any value between the bounds, during the expansion process, we can find an appropriate disclosure set supported in equilibrium where the receiver's ex ante expected utility coincides with that value.

This model, for instance, describes communication between an investor and a consultant. The investor asks professional advice from the consultant who knows the economic environment before she decides whether to invest or not. Imagine a situation where the consultant has a state-dependent bias; that is, the consultant is more eager to invest than the investor when the state is good, but he is more reluctant to invest than her when the state is bad. In this situation, the investor cannot distinguish whether the consultant conceals bad information to induce the investment or conceals good information to deter the investment when receiving ambiguous advice. Our results predict reasonable behaviors in this context. On the one hand, the most informative equilibrium associates with the scenario where the aggressive and the defensive types are pooling by advising same way, e.g., "state is neutral as usual." On the other hand, the least informative equilibrium associates with the scenario where the aggressive and the defensive types adopt different advices, e.g., the aggressive types emphasize only the positive factors, but the defensive types emphasizes only the negative factor.

The paper is organized as follows. In the next subsection, we discuss the related literature. In Section 2, we outline the model. In Section 3, we characterize the set of equilibria. In Section 4, we discuss some extensions. Section 5 concludes the paper.

1.1 Related Literature

The seminal studies of persuasion games are those of Milgrom (1981) and Milgrom and Roberts (1986).[4] These papers assume that (i) the sender's preference is type-independent (e.g., monotonic in the receiver's action), (ii) the receiver can distinguish whether the sender discloses all information, and (iii) no one can commit any strategies. In this environment, full information disclosure can be supported as the unique equilibrium outcome by undertaking the sender's most unfavorable action as a punishment for withholding information. This is the well-known *unraveling argument* in the literature. The subsequent researches revisit the above assumptions and check the validity of the unraveling argument.

Seidmann and Winter (1997), Giovannoni and Seidmann (2007) and Hagenbach et al. (2013) relax assumption (i). Seidmann and Winter (1997) and Giovannoni and Seidmann (2007) assume that the sender's preference is also type-dependent. The players have single-peaked preferences in the receiver's action, and the bliss points vary depending on the sender's private information. These papers show that satisfying the single-crossing condition is the necessary and sufficient condition for full information disclosure in the environment where the players have the single-peaked preferences. Recently, Hagenbach et al. (2013) further relax assumption (i) and analyze a more general environment including monotonic and single-peaked preferences.[5] They show that the acyclicity of mimicking incentives is the necessary and sufficient condition for full information disclosure in the general environment as long as the players have degenerated beliefs off the equilibrium path.[6]

Shin (1994a, 1994b), Lipman and Seppi (1995), Wolinsky (2003) and Mathis (2008) are categorized in the branch relaxing assumption (ii).[7] Because the receiver cannot correctly recognize whether the sender discloses everything, full information disclosure becomes hard to hold. Mathis (2008) characterizes the necessary and sufficient condition for the unraveling argument, which is more demanding compared with that in the fully certifiable environments.

As a departure from full information disclosure, Forges and Koessler (2008) geometrically characterize the set of all Nash and perfect Bayesian equilibria in one-round and multi-round finite persuasion games with assumption (ii). Their characterization is quite general, and the results can apply to any finite persuasion games holding assumption (ii). Lanzi and Mathis (2008) and

[4]The idea of certifiable information disclosure had been used in industrial organization theory before they formalized the concepts. See Grossman (1981) and Grossman and Hart (1980).

[5]We appreciate an anonymous referee for this reference.

[6]Hagenbach et al. (2013) call this condition *acyclic masquerade relation*.

[7]Shin (1994a, 1994b) study the situations where the sender is imperfectly informed. On the other hand, Lipman and Seppi (1995), Wolinsky (2003) and Mathis (2008) analyze the *partially certifiable* environments, in which some private information is certifiable, but the others is not.

Dziuda (2011) characterize the non-full-disclosure behaviors more concretely. Lanzi and Mathis (2008) characterize equilibria in a partially certifiable persuasion game where the receiver has binary alternatives. Dziuda (2011) considers the "strategic argumentation" model, in which the sender's private information represents the "number of arguments" that endorses each alternative, and discusses the properties of equilibria.

Glazer and Rubinstein (2004, 2006) and Kamenica and Gentzkow (2011) study the non-full-disclosure behaviors in the environments where assumption (iii) is relaxed. In Glazer and Rubinstein (2004, 2006), the receiver can commit to an action rule, and they characterize the optimal "persuasion rule" that minimizes the probability that the receiver acts incorrectly. In Kamenica and Gentzkow (2011), on the other hand, the sender chooses his own informativeness before the communication, but he commits to disclose what he knows. They characterize the sender-optimal equilibrium in this environment.

This paper can be regarded as a complement of the above papers. First, this paper is located in the branch of relaxing assumption (i) like Seidmann and Winter (1997), Giovannoni and Seidmann (2007) and Hagenbach et al. (2013). However, the main objective of this paper is characterizing the set of equilibria when full information disclosure is impossible instead of focusing on full information disclosure. Second, because of the cost of generality, the characterization by Forges and Koessler (2008) is quite abstract, and then we still know little about properties of each equilibrium. This paper tries to fill the gap by characterizing the equilibrium set from the viewpoint of the informativeness of each equilibrium.[8] Third, the motivation of this paper is similar to those Lanzi and Mathis (2008) and Dziuda (2011), but the emphasized points are different. Although these papers relax assumptions (ii) to focus on the aspects of the sender's certifiability, this paper relaxes assumption (i) in order to focus on the preference aspect.[9] Finally, we do not allow any commitment of the players as a first step of analysis.

2 The Model

There is one sender and one receiver. The receiver chooses an action $y \in Y \equiv \{y_1, y_2\}$, but the outcome produced by action y_i depends on the sender's private information. Let $\theta \in \Theta \equiv [0, 1]$ be the sender's private information. We interchangeably call set Θ the type space or state space. Let

[8] For simple representation, we adopt a continuum state space model with binary actions instead of finite games adopted in Forges and Koessler (2008).

[9] Our setup is similar to that of Lanzi and Mathis (2008). In their model, the sender's preference satisfy a single-crossing condition by Giovannoni and Seidmann (2007), but the private information is partially certifiable. While we assume full certifiability of the private information, we study a model without the single crossing condition to emphasize the preference aspects.

$F(\cdot)$ be the prior distribution function on the type space Θ with full support density function $f(\cdot)$; that is, $f(\theta) > 0, \forall \theta \in \Theta$.

Let $M(\theta) \equiv \{X \in \mathbb{P}(\Theta) | \theta \in X\}$ be the sender's message space when the sender's type is θ, where $\mathbb{P}(\Theta)$ is the power set of the type space Θ. Any available message must contain the true information θ. Define $M \equiv \cup_{\theta \in \Theta} M(\theta) = \mathbb{P}(\Theta)$, where $m \in M$ represents a message sent by the sender. Note that for any subset $P \subseteq \Theta$, message $m = P$ has a property such that $M^{-1}(P) = P$. That is, this is a *fully certifiable* environment.

We denote the receiver's and the sender's von Neumann–Morgenstern utility functions by $u : \Theta \times Y \to \mathbb{R}$ and $v : \Theta \times Y \to \mathbb{R}$, respectively. We assume that both $u(\theta, y)$ and $v(\theta, y)$ are continuous in θ for any $y \in Y$. Depending on conflicts between the players, the state space is partitioned into the following five regions:

$$\begin{aligned}
\Theta_{11} &\equiv \{\theta \in \Theta | u(\theta, y_1) > u(\theta, y_2) \text{ and } v(\theta, y_1) > v(\theta, y_2)\} \\
\Theta_{22} &\equiv \{\theta \in \Theta | u(\theta, y_2) > u(\theta, y_1) \text{ and } v(\theta, y_2) > v(\theta, y_1)\} \\
\Theta_{12} &\equiv \{\theta \in \Theta | u(\theta, y_1) > u(\theta, y_2) \text{ and } v(\theta, y_2) > v(\theta, y_1)\} \qquad (1) \\
\Theta_{21} &\equiv \{\theta \in \Theta | u(\theta, y_2) > u(\theta, y_1) \text{ and } v(\theta, y_1) > v(\theta, y_2)\} \\
\Theta_0 &\equiv \Theta \backslash (\Theta_{11} \cup \Theta_{22} \cup \Theta_{12} \cup \Theta_{21})
\end{aligned}$$

It is worth mentioning that if θ lies in region $\Theta_{11} \cup \Theta_{22} \cup \Theta_0$, then the sender and the receiver have no conflict. We call regions Θ_{11}, Θ_{22}, and Θ_0 *agreement regions.* On the other hand, if θ lies in region $\Theta_{12} \cup \Theta_{21}$, then there is conflict between the players. That is, if $\theta \in \Theta_{12}$, then the receiver prefers y_1 but the sender prefers y_2. Similarly, if $\theta \in \Theta_{21}$, then the receiver prefers y_2 but the sender prefers y_1. Hence, we call regions Θ_{12} and Θ_{21} *disagreement regions.* To avoid unnecessary complexity, we assume that each region is measurable, and regions $\Theta_{11}, \Theta_{22}, \Theta_{12}$, and Θ_{21} have positive measure but region Θ_0 has zero measure. In other words, the sender and the receiver have nontrivial conflicts.

The timing of the game is as follows. First, nature chooses the state of nature $\theta \in \Theta$ according to the prior distribution $F(\cdot)$. Only the sender observes the state θ. Second, the sender sends a message $m \in M(\theta)$ given the state θ. Then, after observing the message, the receiver undertakes an action $y \in Y$.

The sender's pure strategy $\sigma : \Theta \to M$ specifies a message sent by the sender. The receiver's pure strategy $\mu : M \to Y$ describes an action that she chooses when she observes message m. Let $\mathcal{P} : M \to \Delta(\Theta)$ be the posterior belief of the receiver. This is a function from the entire message space M to the set of probability distributions on the type space Θ.[10]

We use the perfect Bayesian equilibrium (hereafter, PBE) as a solution concept and focus on pure strategy equilibria. Because the information about the state is hard information, any message must contain the true information. In other words, the receiver can infer that the states not included in the observed message never occur for certain. Thus, we must place a restriction on the receiver's equilibrium belief. Letting $S(\mathcal{P}(\cdot|m))$ be the support of the receiver's belief $\mathcal{P}(\cdot|m)$, this requirement is described below.

Requirement 1 *Given a message m, $S(\mathcal{P}(\cdot|m)) \subseteq m$.*

Definition 1 ***PBE***

A triple $(\sigma^*, \mu^*; \mathcal{P}^*)$ is a PBE if it satisfies the following conditions:

(i) $\sigma^*(\theta) \in \arg\max_{m \in M(\theta)} v(\theta, \mu^*(m)), \forall \theta \in \Theta$*;*

(ii) $\mu^*(m) \in \arg\max_{y \in Y} \mathbb{E}_{\mathcal{P}^*(\cdot|m)}[u(\theta, y)], \forall m \in M$*;*

(iii) *$\mathcal{P}^*$ is derived by σ^* consistently from Bayes' rule whenever possible. Otherwise, $\mathcal{P}^*$ is any probability distribution satisfying Requirement 1.*

3 Characterization of Equilibrium Set

3.1 Impossibility of full information disclosure and full information suppression

First, we show that full information disclosure and full information suppression are impossible as a preliminary result. Define $y^R(\theta) \in \arg\max_{y \in Y} u(\theta, y)$.[11] We say that a PBE $(\sigma^*, \mu^*; \mathcal{P}^*)$ is a *full disclosure equilibrium* if $\mu^*(\sigma^*(\theta)) = y^R(\theta)$ for any $\theta \in \Theta$. That is, in the full disclosure equilibrium, the receiver can always undertake her most preferred action. Hereafter, we call action $y^R(\theta)$ *the first-best action*. We say that a PBE $(\sigma^*, \mu^*; \mathcal{P}^*)$ is a *full pooling equilibrium* if $\sigma^*(\theta) = \Theta$ for any $\theta \in \Theta$.

Proposition 1 *There exists neither full disclosure nor full pooling equilibrium.*

All proofs are in the Appendix. These impossibility results are well known in the literature. Because disagreement regions Θ_{12} and Θ_{21} are both nonempty, type $\theta \in \Theta_{12}$ wants to mimic type $\theta' \in \Theta_{21}$,

[10] In relaxed notation, let $\mathcal{P}(\cdot|m)$ represent a conditional probability function if the support of the posterior is countable, and a conditional density function if the support is uncountable.

[11] Note that for any $\theta \in \Theta \backslash \Theta_0$, $y^R(\theta)$ is uniquely determined.

and vice versa.[12] Hence, if the first-best action $y^R(\theta)$ is induced in each state, then at least one of the types has an incentive to deviate to message $m = \{\theta, \theta'\}$ whatever off-the-equilibrium-path beliefs are. That is, the full disclosure equilibrium does not exists. Similarly, because agreement regions Θ_{11} and Θ_{22} are both nonempty, types in these regions prefer disclosing themselves to mimicking other types by pooling. Therefore, the fully pooling behavior is never supported in equilibrium.[13]

3.2 Characterization of equilibrium set

In this subsection, we characterize the set of pure strategy equilibria in terms of informativeness measured by the receiver's equilibrium ex ante expected utility when there exists neither full disclosure nor full pooling equilibrium. We say that equilibrium $(\sigma, \mu; \mathcal{P})$ is *more informative* than equilibrium $(\sigma', \mu'; \mathcal{P}')$ if $\mathbb{E}[u(\theta, \mu(\sigma(\theta)))] \geq \mathbb{E}[u(\theta, \mu'(\sigma'(\theta)))]$, i.e., the former gives higher ex ante expected utility to the receiver. First, we characterize the most and the least informative equilibria and then show that any degree of informativeness between the bounds can be supported in equilibrium with transparent construction of the associated equilibrium. The most informative equilibrium is given by the following proposition.

Proposition 2 *There exists an equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ in which:*

$$\sigma^+(\theta) = \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_{22} \cup \Theta_0 \\ \Theta_{12} \cup \Theta_{21} & \text{if } \theta \in \Theta_{12} \cup \Theta_{21} \end{cases} \tag{2}$$

Moreover, this equilibrium is one of the most informative equilibria.

This is an equilibrium in which types who disagree with the receiver are fully pooling, and other types disclose their own types. Hence, the first-best action $y^R(\theta)$ can be induced in the both agreement regions and one of the disagreement regions. In our environment, each equilibrium is characterized by the set of types in the disagreement regions where the first-best action $y^R(\theta)$ is induced. Hereafter, we call this set *disclosure set*. Hence, the disclosure set of equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ is either disagreement region Θ_{12} or Θ_{21}.[14] Furthermore, this proposition guarantees that we need not consider more complicated partitions of the state space in order to find the best

[12] In other words, our environment violates the single-crossing condition by Giovannoni and Seidmann (2007), and there is a cyclic masquerade relation in the terminology of Hagenbach et al. (2013).

[13] This is a corollary of Proposition 3.3 of Giovannoni and Seidmann (2007); that is, Condition $A1$ is violated in our environment.

[14] The disclosure set in equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ is region Θ_{12} if and only if $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$.

equilibrium for the receiver. In other words, we can conclude that the receiver has to give up at least the amount of information that is equivalent to either disagreement region Θ_{12} or Θ_{21} in any equilibrium.

Intuitively, the reason why equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ is most informative comes from the mutual mimicking incentives of types in the disagreement regions, like the reason for no full information disclosure. As we have mentioned, the disclosure set of equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ is either disagreement region Θ_{12} or Θ_{21}. Hence, in order to dominate this equilibrium, we have to elicit information from types in the complement of the disclosure set. That is, it is necessary that the first-best actions $y^R(\theta)$ and $y^R(\theta')$ are induced for some $\theta \in \Theta_{12}$ and $\theta' \in \Theta_{21}$, simultaneously. However, it is impossible because types θ and θ' have mutual mimicking incentives. Therefore, further improvement of informativeness is impossible, and then equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ is most informative.

Unlike the most informative equilibrium, the characterization of the least informative equilibrium depends crucially on the receiver's utility function and the distribution of θ. Hereafter, to simplify representations, we write $\mathbb{E}[\cdot|Z] = \mathbb{E}[\cdot|\theta \in Z]$. There are the following four cases to be considered:

(1) $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$;

(2) $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] > \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$;

(3) $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$;

(4) $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] > \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$.

The least informative equilibrium is characterized as follows.

Proposition 3 *There exists equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ in which:*

(i) *in Case (1),*

$$\sigma^-(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21} \\ \Theta_{22} \cup \Theta_{12} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12} \\ \{\theta\} & \text{if } \theta \in \Theta_0; \end{cases} \tag{3}$$

(ii) *in Case (2),*

$$\sigma^-(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21} \\ \Theta_{22} \cup \bar{\Theta}_{12} & \text{if } \theta \in \Theta_{22} \cup \bar{\Theta}_{12} \\ \{\theta\} & \text{if } \theta \in (\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta_0; \end{cases} \tag{4}$$

(iii) *in Case (3),*

$$\sigma^-(\theta) = \begin{cases} \Theta_{11} \cup \bar{\Theta}_{21} & \text{if } \theta \in \Theta_{11} \cup \bar{\Theta}_{21} \\ \Theta_{22} \cup \Theta_{12} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12} \\ \{\theta\} & \text{if } \theta \in (\Theta_{21} \backslash \bar{\Theta}_{21}) \cup \Theta_0; \end{cases} \tag{5}$$

(iv) *in Case (4),*

$$\sigma^-(\theta) = \begin{cases} \Theta_{11} \cup \bar{\Theta}_{21} & \text{if } \theta \in \Theta_{11} \cup \bar{\Theta}_{21} \\ \Theta_{22} \cup \bar{\Theta}_{12} & \text{if } \theta \in \Theta_{22} \cup \bar{\Theta}_{12} \\ (\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21}) & \text{if } \theta \in (\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21}) \\ \{\theta\} & \text{if } \theta \in \Theta_0, \end{cases} \tag{6}$$

where $\bar{\Theta}_{12}$ and $\bar{\Theta}_{21}$ are subsets of regions Θ_{12} and Θ_{21} such that $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \bar{\Theta}_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \bar{\Theta}_{21}]$, respectively. Moreover, this is one of the least informative equilibria in each case.

In equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$, types in the disagreement regions are pooling with types in the agreement regions as much as possible. To simplify the exploitation, we focus on Case (2). Intuitively, this is a situation where all types in the disagreement region Θ_{21} can successfully conceal themselves by mimicking types in agreement region Θ_{11}, but it is impossible for types in disagreement region Θ_{12}. That is, some types in disagreement region Θ_{12} can conceal themselves by mimicking types in agreement region Θ_{22}. However, full suppression of disagreement region Θ_{12} by this way is impossible; if all types in region Θ_{12} are pooling with types in region Θ_{22}, then the receiver undertakes action y_1. Types in region Θ_{22} then deviate to full-disclosure message for inducing action y_2.

The structure of equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ of Case (2) is as follows. All types in disagreement region Θ_{21} are pooling with types in agreement region Θ_{11}, and part of types in disagreement region Θ_{12} are pooling with types in agreement region Θ_{22} up to the point where the receive is indifferent between the actions given the pooling types, and the remaining types disclose themselves. Hence, the disclosure set of this equilibrium is region $\Theta_{12} \backslash \bar{\Theta}_{12}$; that is, equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ is characterized by region $\bar{\Theta}_{12}$. Again, this proposition guarantees that it is unnecessary to consider more complicated partition of the state space for constructing the least informative equilibrium; any equilibrium attains higher ex ante expected utility than equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$. It is worthwhile to remark that the structure of region $\bar{\Theta}_{12}$ is irrelevant to the result. Any subset $\tilde{\Theta}_{12}$ of region Θ_{12}

satisfying the condition that $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \tilde{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \tilde{\Theta}_{12}]$ minimizes the receiver's ex ante expected utility.

The reason why equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ attains the minimum is as follows. To be dominated by this equilibrium, some types in disclosure set $\Theta_{12} \backslash \bar{\Theta}_{12}$ should conceal themselves by mimicking other types; let $\Theta'_{12} \subseteq \Theta_{12} \backslash \bar{\Theta}_{12}$ be the set of those types. That is, types in region Θ'_{12} must be pooling with types in region $\Theta_{22} \cup \Theta_{21}$. However, there is no additional room in region Θ_{22} because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$. Hence, the types in region Θ'_{12} must be pooling with types in region Θ_{21}. Let $\Theta'_{21} \subseteq \Theta_{21}$ be the set types in region Θ_{21} that are pooling with types in region Θ'_{12}. Because of the mutual mimicking incentives between types in regions Θ_{12} and Θ_{21}, $\Theta'_{12} = \Theta_{12} \backslash \bar{\Theta}_{12}$ must hold; otherwise, either type $\theta \in \Theta_{12} \backslash (\bar{\Theta}_{12} \cup \Theta'_{12})$ or type $\theta' \in \Theta'_{21}$ has an incentive to deviate to message $m = \{\theta, \theta'\}$. That is, the following condition should hold:

$$\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta'_{21}] \leq \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta'_{21}]. \tag{7}$$

However, (7) means that concealing types in region $\Theta_{12} \backslash \bar{\Theta}_{12}$ by pooling with types in region Θ'_{21} gives weakly better ex ante expected utility to the receiver than equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$. Therefore, we can conclude that equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ is least informative.

There are two implications from these results. First, at least all types in either one of the disagreement regions conceal themselves in any equilibrium. In the most informative equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$, the disclosure set is exactly one of the disagreement regions. That is, in any equilibrium, the receiver has to give up undertaking the first-best action $y^R(\theta)$ over a region including either entire region Θ_{12} or Θ_{21}. Second, as long as the conflict between the players is nontrivial in the sense that either one of Cases (2), (3) and (4) occurs, full suppression of unfavorite information is impossible in any equilibrium. Because the receiver can undertake the first-best action $y^R(\theta)$ over disclosure set $\Theta_{12} \backslash \bar{\Theta}_{12}$ in the least informative equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$, full suppression of such unfavorite information is impossible in any equilibrium under the nontrivial conflict scenarios.

Given the results so far, we can characterize the set of pure strategy equilibria. Define $U^+ \equiv \mathbb{E}[u(\theta, \mu^+(\sigma^+(\theta)))]$ and $U^- \equiv \mathbb{E}[u(\theta, \mu^-(\sigma^-(\theta)))]$. As shown in the following theorem, any value between U^- and U^+ can be equilibrium ex ante expected utility of the receiver. In other words, there are continuum equilibria in this setup.

Theorem 1 *There exists an equilibrium $(\sigma, \mu; \mathcal{P})$ such that $\mathbb{E}[u(\theta, \mu(\sigma(\theta)))] = U$ if and only if $U \in [U^-, U^+]$.*

Intuitively, by continuously shrinking or expanding the disclosure set of an equilibrium, any value between the bounds can be supported as equilibrium ex ante expected utility of the receiver. To simplify the exploitation, we assume that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$ and Case (2). Let X^+ and X^- be the disclosure sets in the most and least informative equilibrium, respectively. That is, $X^+ = \Theta_{12}$ and $X^- = \Theta_{12} \backslash \bar{\Theta}_{12}$ under this assumptions. Because the structure of region $\bar{\Theta}_{12}$ is irrelevant to the result, we define region $\bar{\Theta}_{12}$ as follows: $\bar{\Theta}_{12} \equiv \{\theta \in \Theta_{12} \mid \theta_{12}^- \leq \theta \leq \bar{\delta}\}$ where $\theta_{12}^- \equiv \inf \Theta_{12}$, $\theta_{12}^+ \equiv \sup \Theta_{12}$, and $\bar{\delta} \in (\theta_{12}^-, \theta_{12}^+)$ such that:

$$\int_{\Theta_{22} \cup \{\theta \in \Theta_{12} | \theta_{12}^- \leq \theta \leq \bar{\delta}\}} (u(\theta, y_1) - u(\theta, y_2)) f(\theta) d\theta = 0. \tag{8}$$

By starting from region $\bar{\Theta}_{12}$ and continuously decreasing the upper bound of that region, call the new region $\tilde{\Theta}_{12}$, region $\Theta_{12} \backslash \tilde{\Theta}_{12}$ continuously expands, and finally converges to disclosure set X^+. Hence, for any $U' \in [U^-, U^+]$, we can find an appropriate set $X' \supseteq X^-$ through the process of expansion such that if the receiver undertakes the first-best action $y^R(\theta)$ on X', then her ex ante expected utility is U'. We can easily show that this partition can be supported in an equilibrium.

This expansion argument can be easily extended to the situation where disclosure sets X^+ and X^- are included in different disagreement regions. We assume that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$ and Case (3). That is, $X^+ = \Theta_{12}$ and $X^- = \Theta_{21} \backslash \bar{\Theta}_{21}$. In this scenario, there exists a threshold $\hat{U} \in [U^-, U^+]$ such that ex ante expected utility $\hat{U}$ is supported by both disclosure sets $\hat{X}_{21} \subseteq \Theta_{21}$ and $\hat{X}_{12} \subseteq \Theta_{12}$. Define $\hat{X}_{21} = \Theta_{21}$ and $\hat{X}_{12} = \Theta_{12} \backslash \hat{\Theta}_{12}$ where $\hat{\Theta}_{12} \equiv \{\theta \in \Theta_{12} | \theta_{12}^- \leq \theta \leq \hat{\delta}\}$ for $\hat{\delta} \in [\theta_{12}^-, \theta_{12}^+)$ such that:

$$\int_{(\Theta_{12} \backslash \{\theta \in \Theta_{12} | \theta_{12}^- \leq \theta \leq \hat{\delta}\}) \cup \Theta_{21}} (u(\theta, y_1) - u(\theta, y_2)) f(\theta) d\theta = 0. \tag{9}$$

It is easily shown that ex ante expected utility associated with disclosure sets $\hat{X}_{21}$ and $\hat{X}_{12}$ coincides, and these sets are supported in equilibrium. Hence, we can divide interval $[U^-, U^+]$ into two subintervals $[U^-, \hat{U}]$ and $[\hat{U}, U^+]$, and separately apply the expansion argument to each interval. In other words, for any value $U \in [U^-, \hat{U}]$, we can find an appropriate set X where $\hat{X}_{21} \supseteq X \supseteq X^-$ and the associated ex ante expected utility is U. Furthermore, for any value $U' \in [\hat{U}, U^+]$, we can also find an appropriate set X' where $X^+ \supseteq X' \supseteq \hat{X}_{12}$ and the associated ex ante expected utility is U'. Therefore, we can conclude that $[U^-, U^+]$ is the set of equilibrium ex ante expected utility of the receiver.

4 Discussion

4.1 Full disclosure and full pooling equilibria

The characterization of the equilibrium set can be extended to the environment where there exists full disclosure or full pooling equilibrium. So far, we have focused on the environment where these extreme equilibria do not exist; that is, any Θ_{ij} has positive measure. However, as long as the receiver's action is binary and any information is fully certifiable, the same characterization is still valid even in the environment where the full disclosure or full pooling equilibrium exists; that is, some Θ_{ij} has zero measure. In other words, the ex ante expected utility in the full disclosure and full pooling equilibria is equivalent to that obtained in equilibria specified in the previous propositions. Hence, by the same argument, Theorem 1 can extend to the environment where these extreme equilibria exist.

Corollary 1 *Suppose that some Θ_{ij} has zero measure, i.e., there exists either full disclosure or full pooling equilibrium. Then, Theorem 1 holds in this modified environment.*

4.2 Mixed strategy equilibria

The characterization of the equilibrium set can be partially extended to the scenario where the players could adopt mixed strategies. We put the following additional restrictions.

Assumption 1

(i) *Suppose that* $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \neq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$ *holds.*

(ii) *Suppose that either* $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ *or* $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$ *holds.*[15]

Corollary 2 *Suppose that Assumption 1 holds. Then, Theorem 1 holds.*

The outline of the proof is as follows. Under Assumption 1, we can show that the ex ante expected utility of any mixed strategy equilibrium where the receiver adopts a mixed strategy is bounded by the bounds characterized so far. That is, for any such mixed strategy equilibrium, there exists a pure strategy equilibrium that is payoff equivalent to that mixed strategy equilibrium. Hence, without loss of generality, we can restrict our attention to equilibria where the receiver adopts pure strategies. Furthermore, note that the proofs in the propositions and the theorem

[15] This assumption excludes Case (4) in the characterization of the least informative equilibrium.

do not depend on whether the sender adopts pure strategies. Therefore, once we can show the redundancy of considering the receiver's mixed strategies, all proofs go through, and then Corollary 2 is obtained.

5 Conclusion

In this paper, we have analyzed a persuasion game where the receiver's alternatives are binary. Especially, we have focused on the scenario where there exists neither full disclosure nor full pooling equilibrium and have characterized the set of pure strategy in terms of informativeness measured by ex ante expected utility of the receiver. In other words, we have characterized the informativeness of each equilibrium. The set is characterized by the most and least informative equilibria, and we have shown that any value between the bounds can be supported in equilibrium with transparent construction of the associated equilibrium. That is, there are continuum equilibria. The results depend on the restriction to a setting with binary actions, and that extension is left for the future work.

Appendix: Proofs

Proof of Proposition 1

Lemma 1 *In any equilibrium,* $\mu(\sigma(\theta)) = y^R(\theta)$ *for all* $\theta \in \Theta_{11} \cup \Theta_{22}$.

Proof of Lemma 1. Suppose, by contrast, that there exists an equilibrium $(\sigma, \mu; \mathcal{P})$, in which there exists a type $\theta \in \Theta_{11} \cup \Theta_{22}$ such that $\mu(\sigma(\theta)) \neq y^R(\theta)$. However, this type has the same preference as the receiver, and he can prove his true type by sending the message $m = \{\theta\}$. So, this message induces the sender's preferred action; that is, it is a profitable deviation for him, which is a contradiction. Therefore, these types must induce preferred actions in any equilibrium. ■

Proof of Proposition 1. First, show that there exists no full disclosure equilibrium. Suppose, by contrast, that there exists a full disclosure equilibrium $(\sigma^*, \mu^*; \mathcal{P}^*)$. Because $\Theta_{12} \neq \emptyset$ and $\Theta_{21} \neq \emptyset$, pick arbitrary types $\theta \in \Theta_{12}$ and $\theta' \in \Theta_{21}$. Note that the type θ strictly prefers action y_2 to action y_1, and the type θ' strictly prefers action y_1 to action y_2. Because $(\sigma^*, \mu^*; \mathcal{P}^*)$ is a full disclosure equilibrium, $\mu^*(\sigma^*(\theta)) = y_1$ and $\mu^*(\sigma^*(\theta')) = y_2$. However, there is no incentive-compatible reaction to message $\{\theta, \theta'\} \in M(\theta) \cap M(\theta')$; if $\mu^*(\{\theta, \theta'\}) = y_1$, then type θ' has an incentive to deviate, and if $\mu^*(\{\theta, \theta'\}) = y_2$, then type θ has an incentive to deviate, which is a contradiction. Thus, there exists no full disclosure equilibrium.[16]

Next, show that there exists no full pooling equilibrium. Suppose, by contrast, that there exists a full pooling equilibrium $(\sigma^*, \mu^*; \mathcal{P}^*)$. That is, $\sigma^*(\theta) = \Theta$ for any $\theta \in \Theta$. However, by Lemma 1, types in region $\Theta_{11} \cup \Theta_{22}$ have an incentive to deviate to the full-disclosure message, which is a contradiction. Therefore, there exists no full pooling equilibrium. ■

Proof of Proposition 2

We assume that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$ without loss of generality. Hence, $\mu^+(\Theta_{12} \cup \Theta_{21}) = y_1$. We omit the characterizations of PBEs and the related proofs. These are in the Supplementary Appendix.[17] We show that equilibrium $(\sigma^+, \mu^+; \mathcal{P}^+)$ attains the maximum ex ante expected utility to the receiver. Suppose, by contrast, that there exists an equilibrium $(\hat{\sigma}, \hat{\mu}; \hat{\mathcal{P}})$ such that:

$$\mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))] > \mathbb{E}[u(\theta, \mu^+(\sigma^+(\theta)))]. \tag{10}$$

By (10) and Lemma 1:

$$\begin{aligned}
&\mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\qquad + \mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))|\Theta_0]Pr(\Theta_0) + \mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{12} \cup \Theta_{21}]Pr(\Theta_{12} \cup \Theta_{21}) \\
> &\mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\qquad + \mathbb{E}[u(\theta, \mu^+(\sigma^+(\theta)))|\Theta_0]Pr(\Theta_0) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}]Pr(\Theta_{12} \cup \Theta_{21}),
\end{aligned}$$

or:

$$\mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{12} \cup \Theta_{21}] > \mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}]. \tag{11}$$

Define $W \equiv \{\theta \in \Theta_{21} | \hat{\mu}(\hat{\sigma}(\theta)) = y_2\}$, and $Z \equiv \{\theta \in \Theta_{12} | \hat{\mu}(\hat{\sigma}(\theta)) = y_1\}$.

Claim 1 *If equation (11) holds, then $W \neq \emptyset$.*

Proof of Claim 1. Suppose, by contrast, that $W = \emptyset$. That is, for any $\theta \in \Theta_{21}$, $\hat{\mu}(\hat{\sigma}(\theta)) = y_1$. By (11):

$$\begin{aligned}
&\mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12} \cup \Theta_{21}) + \mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{21}]Pr(\Theta_{21}|\Theta_{12} \cup \Theta_{21}) \\
&\qquad > \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12} \cup \Theta_{21}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}|\Theta_{12} \cup \Theta_{21}).
\end{aligned}$$

[16]It is worthwhile to mention that this part of the proof is valid even if mixed strategies are allowed.
[17]Supplementary Appendix is available from the author's homepage, `http://smiura.web.fc2.com/files/persuasion_suppapp.pdf`.

From the hypothesis:

$$\begin{aligned}
&\mathbb{E}[u(\theta,\hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_1)|\Theta_{21}]Pr(\Theta_{21}|\Theta_{12}\cup\Theta_{21})\\
&\quad>\mathbb{E}[u(\theta,y_1)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_1)|\Theta_{21}]Pr(\Theta_{21}|\Theta_{12}\cup\Theta_{21}),
\end{aligned}$$

or:

$$\mathbb{E}[u(\theta,\hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{12}]>\mathbb{E}[u(\theta,y_1)|\Theta_{12}]. \tag{12}$$

However, as long as $\theta\in\Theta_{12}, u(\theta,y_1)>u(\theta,y_2)$. Hence, $\mathbb{E}[u(\theta,y_1)|\Theta_{12}]\geq\mathbb{E}[u(\theta,\hat{\mu}(\hat{\sigma}(\theta)))|\Theta_{12}]$, which is a contradiction to (12). $\square$

Claim 2 *If equation (11) holds, then $Z\neq\emptyset$.*

Proof of Claim 2. Suppose, by contrast, that $Z=\emptyset$; that is, for all $\theta\in\Theta_{12}$, $\hat{\mu}(\hat{\sigma}(\theta))=y_2$. By (11), the hypothesis and the definition of W:

$$\begin{aligned}
&\mathbb{E}[u(\theta,y_2)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_2)|W]Pr(W|\Theta_{12}\cup\Theta_{21})\\
&\qquad\qquad+\mathbb{E}[u(\theta,y_1)|\Theta_{12}\backslash W]Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21})\\
&>\mathbb{E}[u(\theta,y_1)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_1)|W]Pr(W|\Theta_{12}\cup\Theta_{21})\\
&\qquad\qquad+\mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash W]Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21}),
\end{aligned}$$

or:

$$\begin{aligned}
&\mathbb{E}[u(\theta,y_2)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_2)|W]Pr(W|\Theta_{12}\cup\Theta_{21})\\
&\quad>\mathbb{E}[u(\theta,y_1)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_1)|W]Pr(W|\Theta_{12}\cup\Theta_{21}).
\end{aligned}\tag{13}$$

Because $\mathbb{E}[u(\theta,y_1)|\Theta_{12}\cup\Theta_{21}]\geq\mathbb{E}[u(\theta,y_2)|\Theta_{12}\cup\Theta_{21}]$:

$$\begin{aligned}
&\mathbb{E}[u(\theta,y_1)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_1)|W]Pr(W|\Theta_{12}\cup\Theta_{21})\\
&\qquad\qquad+\mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash W]Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21})\\
&\geq\mathbb{E}[u(\theta,y_2)|\Theta_{12}]Pr(\Theta_{12}|\Theta_{12}\cup\Theta_{21})+\mathbb{E}[u(\theta,y_2)|W]Pr(W|\Theta_{12}\cup\Theta_{21})\\
&\qquad\qquad+\mathbb{E}[u(\theta,y_2)|\Theta_{21}\backslash W]Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21}).
\end{aligned}\tag{14}$$

By (13) and (14), the following condition must hold:

$$\mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash W]Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21}) > \mathbb{E}[u(\theta,y_2)|\Theta_{21}\backslash W]Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21}).$$

If $Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21}) = 0$, then, the above inequality does not hold, which is a contradiction. Hence, $Pr(\Theta_{21}\backslash W|\Theta_{12}\cup\Theta_{21}) \neq 0$. That is, $\mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash W] > \mathbb{E}[u(\theta,y_2)|\Theta_{21}\backslash W]$. However, $u(\theta,y_2) > u(\theta,y_1)$ for any $\theta \in \Theta_{21}$. Hence, $\mathbb{E}[u(\theta,y_2)|\Theta_{21}\backslash W] > \mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash W]$, which is a contradiction. □

By Claims 1 and 2, to hold (11), $W \neq \emptyset$ and $Z \neq \emptyset$; that is, there exists $\theta \in \Theta_{12}$ and $\theta' \in \Theta_{21}$ such that $\hat{\mu}(\hat{\sigma}(\theta)) = y_1$ and $\hat{\mu}(\hat{\sigma}(\theta')) = y_2$. However, there is no incentive-compatible reaction to message $\{\theta,\theta'\} \in M(\theta)\cap M(\theta')$; if $\hat{\mu}(\{\theta,\theta'\}) = y_1$, then type θ' has an incentive to deviate, and if $\hat{\mu}(\{\theta,\theta'\}) = y_2$, then type θ has an incentive to deviate, which is a contradiction. Therefore, an equilibrium satisfying (10) cannot exist; that is, equilibrium $(\sigma^+,\mu^+;\mathcal{P}^+)$ is most informative. ■

Proof of Proposition 3

Define $X \equiv \{\theta \in \Theta_{12}\cup\Theta_{21}|\mu(\sigma(\theta)) = y^R(\theta)\}$ given an equilibrium $(\sigma,\mu;\mathcal{P})$. This is the set of states in the disagreement regions where the receiver undertakes her preferred action $y^R(\theta)$.

Lemma 2 *Suppose that either* $\mathbb{E}[u(\theta,y_1)|\Theta_{11}\cup\Theta_{21}] < \mathbb{E}[u(\theta,y_2)|\Theta_{11}\cup\Theta_{21}]$ *or* $\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\Theta_{12}] > \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\Theta_{12}]$. *Then, for any equilibrium* $(\sigma,\mu;\mathcal{P})$, $X \neq \emptyset$, *and either* $X \subseteq \Theta_{12}$ *or* $X \subseteq \Theta_{22}$.

Proof of Lemma 2. Suppose, by contrast, that there exists an equilibrium $(\tilde{\sigma},\tilde{\mu};\tilde{\mathcal{P}})$ such that $\tilde{X} = \emptyset$. That is, for all $\theta \in \Theta_{12}\cup\Theta_{21}$, $\tilde{\mu}(\tilde{\sigma}(\theta)) \neq y^R(\theta)$. In the equilibrium, types in region Θ_{12} are never pooling with types in Θ_{21}; otherwise, there exists a type $\theta \in \Theta_{12}\cup\Theta_{21}$ such that $\tilde{\mu}(\tilde{\sigma}(\theta)) = y^R(\theta)$. Hence, by Lemma 1, it is necessary that (i) each type in region Θ_{12} must be pooling with some types in region $\Theta_{22}\cup\Theta_0$ with inducing action y_2; and (ii) each type in region Θ_{21} must be pooling with some types in region $\Theta_{11}\cup\Theta_0$ with inducing action y_1.

Claim 3 *If condition (i) (resp. (ii)) holds, then* $\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\Theta_{12}\cup\Theta_0] \leq \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\Theta_{12}\cup\Theta_0]$ *(resp.* $\mathbb{E}[u(\theta,y_1)|\Theta_{11}\cup\Theta_{21}\cup\Theta_0] \geq \mathbb{E}[u(\theta,y_2)|\Theta_{11}\cup\Theta_{21}\cup\Theta_0]$*) must hold.*

Proof of Claim 3. Suppose that condition (i) holds. Then there must exist a partition Π of region $\Theta_{22}\cup\Theta_{12}\cup\Theta_0$ such that: (a) for any $P \in \Pi$, if $P\cap\Theta_{12} \neq \emptyset$, then either $P\cap\Theta_{22} \neq \emptyset$ or $P\cap\Theta_0 \neq \emptyset$; and (b) $\mathbb{E}[u(\theta,y_1)|P] \leq \mathbb{E}[u(\theta,y_2)|P]$ for any $P \in \Pi$. Suppose that Π is countable. By properties

(a) and (b) of partition Π, $\mathbb{E}[u(\theta,y_1)|P]Pr(P|\Theta_{22}\cup\Theta_{12}\cup\Theta_0) \le \mathbb{E}[u(\theta,y_2)|P]Pr(P|\Theta_{22}\cup\Theta_{12}\cup\Theta_0)$ for any $P \in \Pi$. Then:

$$\begin{aligned}
\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\Theta_{12}\cup\Theta_0] &= \sum_{P\in\Pi}\mathbb{E}[u(\theta,y_1)|P]Pr(P|\Theta_{22}\cup\Theta_{12}\cup\Theta_0) \\
&\le \sum_{P\in\Pi}\mathbb{E}[u(\theta,y_2)|P]Pr(P|\Theta_{22}\cup\Theta_{12}\cup\Theta_0) \\
&= \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\Theta_{12}\cup\Theta_0].
\end{aligned}$$

For the scenario where partition Π is uncountable, we can show the claim by the similar argument. Furthermore, by using the same argument, we can show that condition (ii) implies that $\mathbb{E}[u(\theta,y_1)|\Theta_{11}\cup\Theta_{21}\cup\Theta_0] \ge \mathbb{E}[u(\theta,y_2)|\Theta_{11}\cup\Theta_{21}\cup\Theta_0]$. □

By Claim 3 and $Pr(\Theta_0)=0$, we can conclude that $\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\Theta_{12}] \le \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\Theta_{12}]$ and $\mathbb{E}[u(\theta,y_1)|\Theta_{11}\cup\Theta_{21}] \ge \mathbb{E}[u(\theta,y_2)|\Theta_{11}\cup\Theta_{21}]$ must hold. However, this contradicts with the hypothesis that: $\mathbb{E}[u(\theta,y_1)|\Theta_{11}\cup\Theta_{21}] < \mathbb{E}[u(\theta,y_2)|\Theta_{11}\cup\Theta_{21}]$ or $\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\Theta_{12}] > \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\Theta_{12}]$. Therefore, subset X should be nonempty for any equilibrium.

Next, suppose, by contrast, that there exists an equilibrium $(\hat{\sigma},\hat{\mu};\hat{\mathcal{P}})$ such that $\hat{X}\cap\Theta_{12}\neq\emptyset$ and $\hat{X}\cap\Theta_{21}\neq\emptyset$. Choose $\theta\in\hat{X}\cap\Theta_{12}$ and $\theta'\in\hat{X}\cap\Theta_{21}$ arbitrarily. However, there is no incentive-compatible reaction to message $\{\theta,\theta'\}\in M(\theta)\cap M(\theta')$; if $\hat{\mu}(\{\theta,\theta'\})=y_1$, then type θ' has an incentive to deviate, and if $\hat{\mu}(\{\theta,\theta'\})=y_2$, then type θ has an incentive to deviate, which is a contradiction. Therefore, either $X\subseteq\Theta_{12}$ or $X\subseteq\Theta_{21}$ must hold. ■

Proof of Proposition 3. We omit the characterizations of PBE $(\sigma^-,\mu^-;\mathcal{P}^-)$ and the related proofs. These are in the Supplementary Appendix. Now, we show that this equilibrium attains the minimum ex ante expected utility to the receiver in each case. The proof is constructed by the following lemmas.

Lemma 3 *Equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$ specified by (3) is least informative in Case (1).*

Proof of Lemma 3. It is obvious that equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$ is the least informative equilibrium because the first-best action $y^R(\theta)$ cannot be induced in entire disagreement region $\Theta_{12}\cup\Theta_{21}$. ■

Lemma 4 *Equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$ specified by (4) is least informative in Case (2).*

Proof of Lemma 4. This lemma is shown by the following steps.

Step 1: Construction of subset $\bar{\Theta}_{12}$.

Define $\theta_{12}^{-} \equiv \inf \Theta_{12}$ and $\theta_{12}^{+} \equiv \sup \Theta_{12}$, and then define $\Theta_{12}^{\delta} \equiv \{\theta \in \Theta_{12} \cup \{\theta_{12}^{-}, \theta_{12}^{+}\} | \theta_{12}^{-} \leq \theta \leq \delta\}$ for $\delta \in [\theta_{12}^{-}, \theta_{12}^{+}]$. Define the following function:

$$G(\delta) \equiv \int_{\Theta_{22} \cup \Theta_{12}^{\delta}} (u(\theta, y_1) - u(\theta, y_2)) f(\theta) d\theta.$$

It is clear that $G(\cdot)$ is continuous in δ. Note that $G(\theta_{12}^{+}) = \mathbb{E}[u(\theta, y_1) - u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}] Pr(\Theta_{22} \cup \Theta_{12})$. Because $\mathbb{E}[u(\theta, y_1) | \Theta_{22} \cup \Theta_{12}] > \mathbb{E}[u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}]$ and $Pr(\Theta_{22} \cup \Theta_{12}) > 0$, $G(\theta_{12}^{+}) > 0$. Also note $G(\theta_{12}^{-}) = \mathbb{E}[u(\theta, y_1) - u(\theta, y_2) | \Theta_{22}] Pr(\Theta_{22})$. Because $u(\theta, y_1) < u(\theta, y_2)$ for any $\theta \in \Theta_{22}$ and $Pr(\Theta_{22}) > 0$, $G(\theta_{12}^{-}) < 0$. Therefore, from the intermediate value theorem, there exists $\bar{\delta} \in (\theta_{12}^{-}, \theta_{12}^{+})$ such that $G(\bar{\delta}) = 0$. That is, $\mathbb{E}[u(\theta, y_1) - u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}^{\bar{\delta}}] Pr(\Theta_{22} \cup \Theta_{12}^{\bar{\delta}}) = 0$. Because $Pr(\Theta_{22} \cup \Theta_{12}^{\bar{\delta}}) > 0$, that is equivalent to $\mathbb{E}[u(\theta, y_1) | \Theta_{22} \cup \Theta_{12}^{\bar{\delta}}] = \mathbb{E}[u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}^{\bar{\delta}}]$. Define $\bar{\Theta}_{12} \equiv \Theta_{12}^{\bar{\delta}}$.

Because of Lemma 2, it is sufficient to show that: (i) there exists no equilibrium $(\sigma', \mu'; \mathcal{P}')$ with $X' \subseteq \Theta_{12}$ such that $\mathbb{E}[u(\theta, \mu'(\sigma'(\theta)))] < \mathbb{E}[u(\theta, \mu^{-}(\sigma^{-}(\theta)))]$; and (ii) there exists no equilibrium $(\sigma'', \mu''; \mathcal{P}'')$ with $X'' \subseteq \Theta_{21}$ such that $\mathbb{E}[u(\theta, \mu''(\sigma''(\theta)))] < \mathbb{E}[u(\theta, \mu^{-}(\sigma^{-}(\theta)))]$.

Step 2: Show Condition (i).

Let $(\sigma', \mu'; \mathcal{P}')$ be an arbitrary equilibrium with $X' \subseteq \Theta_{12}$. Define $\Theta_{12}' \equiv \Theta_{12} \backslash X'$. By Lemma 2, $X' \neq \emptyset$ and $X' \cap \Theta_{21} = \emptyset$. Hence, we know that: for any equilibrium $(\sigma', \mu'; \mathcal{P}')$ with $X' \subseteq \Theta_{12}$, (a) $\mu'(\sigma'(\theta)) = y_1$ for any $\theta \in \Theta_{11} \cup \Theta_{21} \cup (\Theta_{12} \backslash \Theta_{12}')$; and (b) $\mu'(\sigma'(\theta)) = y_2$ for any $\theta \in \Theta_{22} \cup \Theta_{12}'$. Let $(\hat{\sigma}, \hat{\mu}; \hat{\mathcal{P}})$ be another equilibrium with $\hat{X} \subseteq \Theta_{12}$, and $\hat{\Theta}_{12} \equiv \Theta_{12} \backslash \hat{X}$. Then, $\mathbb{E}[u(\theta, \mu'(\sigma'(\theta)))] \leq \mathbb{E}[u(\theta, \hat{\mu}(\hat{\sigma}(\theta)))]$ is equivalent to:

$$\begin{aligned} &\mathbb{E}[u(\theta, y_2) | \Theta_{12}'] Pr(\Theta_{12}') + \mathbb{E}[u(\theta, y_1) | \Theta_{12} \backslash \Theta_{12}'] Pr(\Theta_{12} \backslash \Theta_{12}') \\ &\qquad \leq \mathbb{E}[u(\theta, y_2) | \hat{\Theta}_{12}] Pr(\hat{\Theta}_{12}) + \mathbb{E}[u(\theta, y_1) | \Theta_{12} \backslash \hat{\Theta}_{12}] Pr(\Theta_{12} \backslash \hat{\Theta}_{12}) \end{aligned} \tag{15}$$

The following claim represents another restriction that equilibrium $(\sigma', \mu'; \mathcal{P}')$ must satisfy.

Claim 4 $\mathbb{E}[u(\theta, y_1) | \Theta_{22} \cup \Theta_{12}'] \leq \mathbb{E}[u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}']$.

Proof of Claim 4. Suppose, by contrast, that $\mathbb{E}[u(\theta, y_1) | \Theta_{22} \cup \Theta_{12}'] > \mathbb{E}[u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}']$. Note that $\Theta_{12}' = \{\theta \in \Theta_{12} | \mu'(\sigma'(\theta)) = y_2\}$. By the hypothesis, some types in set Θ_{12}' must be pooling with types in region Θ_{21} with inducing action y_2; otherwise, i.e., if each type in region Θ_{12}' is pooling with some types in region $\Theta_{22} \cup \Theta_0$ with inducing y_2, then $\mathbb{E}[u(\theta, y_1) | \Theta_{22} \cup \Theta_{12}'] \leq \mathbb{E}[u(\theta, y_2) | \Theta_{22} \cup \Theta_{12}']$ by the similar arguments in Claim 3, which is a contradiction to the hypothesis.

Let $\theta' \in \Theta_{21}$ be the type that is pooling with some type in Θ'_{12} and inducing action y_2, i.e., $\mu'(\sigma'(\theta')) = y_2$ That is, $\theta' \in X' \cap \Theta_{21}$. However, because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] > \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$ and $X' \subseteq \Theta_{12}$, $X' \cap \Theta_{21} = \emptyset$ by Lemma 2, which is a contradiction. □

By (15) and Claim 4, in order to show Condition (i), it is sufficient to show that $\bar{\Theta}_{12}$ is a solution of the following optimization problem:

$$\begin{aligned} \min_{\Theta'_{12} \subseteq \Theta_{12}} \ & \mathbb{E}[u(\theta, y_2)|\Theta'_{12}]Pr(\Theta'_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \backslash \Theta'_{12}]Pr(\Theta_{12} \backslash \Theta'_{12}) \\ & \text{subject to } \mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta'_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta'_{12}]. \end{aligned} \tag{16}$$

Note that the objective function of (16) is decreasing in the probability measure on Θ'_{12} because $u(\theta, y_1) > u(\theta, y_2)$ for any $\theta \in \Theta_{12}$.

Claim 5 *At the solution of optimization problem (16), the constraint is binding.*

Proof of Claim 5. Suppose, by contrast, that $\hat{\Theta}_{12}$ is a solution of (16) but $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \hat{\Theta}_{12}] < \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \hat{\Theta}_{12}]$. Because of the continuity of the utility function, we can find $\tilde{\Theta}_{12} \supset \hat{\Theta}_{12}$ such that $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \tilde{\Theta}_{12}] < \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \tilde{\Theta}_{12}]$ and $Pr(\tilde{\Theta}_{12}) > Pr(\hat{\Theta}_{12})$. However, because of the monotonicity of the objective function:

$$\begin{aligned} & \mathbb{E}[u(\theta, y_2)|\tilde{\Theta}_{12}]Pr(\tilde{\Theta}_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \backslash \tilde{\Theta}_{12}]Pr(\Theta_{12} \backslash \tilde{\Theta}_{12}) \\ & \qquad < \mathbb{E}[u(\theta, y_2)|\hat{\Theta}_{12}]Pr(\hat{\Theta}_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \backslash \hat{\Theta}_{12}]Pr(\Theta_{12} \backslash \hat{\Theta}_{12}) \end{aligned}$$

This is a contradiction to that $\hat{\Theta}_{12}$ is a solution of (16). □

Now, we show that $\bar{\Theta}_{12}$ is a solution to the problem. The following claim guarantees that subset Θ'_{12} binding the constraint becomes a solution of the problem.

Claim 6 *If Θ'_{12} is a subset of region Θ_{12} such that $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta'_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta'_{12}]$, then Θ'_{12} is a solution of the optimization problem (16).*

Proof of Claim 6. Suppose that subset Θ'_{12} of region Θ_{12} satisfies $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta'_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta'_{12}]$. Then, by Claim 5, subset Θ'_{12} is a candidate of the solution. Now, it is sufficient to show that for any subset of region Θ_{12} binding the constraint, the value of the objective function is identical. Let Θ''_{12} be another subset of region Θ_{12} satisfying $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta''_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta''_{12}]$. Because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta'_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta'_{12}]$, $\mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta_{22} \cup \Theta'_{12}]Pr(\Theta_{22} \cup \Theta'_{12}) = 0$, or still:

$$\mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta'_{12}]Pr(\Theta'_{12}) = 0. \tag{17}$$

Similarly, $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta''_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta''_{12}]$ implies:

$$\mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta''_{12}]Pr(\Theta''_{12}) = 0. \tag{18}$$

By using (17) and (18), we obtain $\mathbb{E}[u(\theta, y_2)-u(\theta, y_1)|\Theta'_{12}]Pr(\Theta'_{12}) = \mathbb{E}[u(\theta, y_2)-u(\theta, y_1)|\Theta''_{12}]Pr(\Theta'_{12})$. Then:

$$\begin{aligned}&\mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta'_{12}]Pr(\Theta'_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12}) \\ &\quad = \mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta''_{12}]Pr(\Theta''_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12})\end{aligned} \tag{19}$$

Note that:

$$\begin{aligned}&\mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta'_{12}]Pr(\Theta'_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12}) \\ = \;&\mathbb{E}[u(\theta, y_2)|\Theta'_{12}]Pr(\Theta'_{12}) - \mathbb{E}[u(\theta, y_1)|\Theta'_{12}]Pr(\Theta'_{12}) \\ &\quad +\mathbb{E}[u(\theta, y_1)|\Theta'_{12}]Pr(\Theta'_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}\backslash\Theta'_{12}]Pr(\Theta_{12}\backslash\Theta'_{12}) \\ = \;&\mathbb{E}[u(\theta, y_2)|\Theta'_{12}]Pr(\Theta'_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}\backslash\Theta'_{12}]Pr(\Theta_{12}\backslash\Theta'_{12}).\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta''_{12}]Pr(\Theta''_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12}) \\ &\quad = \mathbb{E}[u(\theta, y_2)|\Theta''_{12}]Pr(\Theta''_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}\backslash\Theta''_{12}]Pr(\Theta_{12}\backslash\Theta''_{12}).\end{aligned}$$

Therefore, by (19), we can conclude that the values of the objective function evaluated under Θ'_{12} and Θ''_{12} are identical. $\square$

Thus, by Claim 6, subset $\bar{\Theta}_{12}$ is a solution of the optimization problem (16). That is, any equilibrium with $X \subseteq \Theta_{12}$ attains weakly better ex ante expected utility to the receiver than equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$.

Step 3: Show Condition (ii).

Suppose, by contrast, that there exists an equilibrium $(\sigma'', \mu''; \mathcal{P}'')$ with $X'' \subseteq \Theta_{21}$ such that $\mathbb{E}[u(\theta, \mu''(\sigma''(\theta)))] < \mathbb{E}[u(\theta, \mu^-(\sigma^-(\theta)))]$. By Lemma 2, $X'' \cap \Theta_{12} = \emptyset$. Hence, for all $\theta \in \Theta_{12}$, $\mu''(\sigma''(\theta)) = y_2$. That is, each type in region Θ_{12} must be pooling with types in region $\Theta_{22} \cup X'' \cup \Theta_0$. Then, we can partition region Θ_{12} into $\Theta_{12}^1, \Theta_{12}^2$ and Θ_{12}^3 given by:

$$\begin{aligned}
\Theta_{12}^1 &\equiv \{\theta \in \Theta_{12} | \theta \text{ is pooling with types in } \Theta_{22}\} \\
\Theta_{12}^2 &\equiv \{\theta \in \Theta_{12} | \theta \text{ is pooling with types in } X'' \subseteq \Theta_{21}\} \\
\Theta_{12}^3 &\equiv \{\theta \in \Theta_{12} | \theta \text{ is pooling with types in } \Theta_0\}
\end{aligned}$$

Note that $Pr(\Theta_{12}^3) = 0$ because $u(\theta, y_1) > u(\theta, y_2)$ for any $\theta \in \Theta_{12}$ and $Pr(\Theta_0) = 0$. Then, we can say that $Pr(\Theta_{12}^2) > 0$; otherwise, almost all types in region Θ_{12} must be pooling with types in region Θ_{22}, and then $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$ must hold by the similar argument in Claim 3, which is impossible. Hence, $Pr(X'') > 0$ because $Pr(\Theta_{12}^2) > 0$ and $u(\theta, y_1) > u(\theta, y_2)$ for any $\theta \in \Theta_{12}$.

By the similar argument in Claim 3, we obtain:

$$\begin{aligned}
\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^1] &\leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^1]; \quad (20) \\
\mathbb{E}[u(\theta, y_1)|X'' \cup \Theta_{12}^2] &\leq \mathbb{E}[u(\theta, y_2)|X'' \cup \Theta_{12}^2]. \quad (21)
\end{aligned}$$

Multiplying both sides of (20) and (21) by $Pr(\Theta_{22} \cup \Theta_{12}^1 | \Theta_{22} \cup \Theta_{12} \cup X'')$ and $Pr(X'' \cup \Theta_{12}^2 | \Theta_{22} \cup \Theta_{12} \cup X'')$, respectively, and combining these equations yields:

$$\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12} \cup X''] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12} \cup X'']. \quad (22)$$

Claim 7 *Let $(\sigma, \mu; \mathcal{P})$ be an equilibrium with $X \subseteq \Theta_{12}$ and $Pr(X) \neq 0$, and $(\tilde{\sigma}, \tilde{\mu}, \tilde{\mathcal{P}})$ be an equilibrium with $\tilde{X} \subseteq \Theta_{21}$ and $Pr(\tilde{X}) \neq 0$. Then, $\mathbb{E}[u(\theta, \mu(\sigma(\theta)))] \geq \mathbb{E}[u(\theta, \tilde{\mu}(\tilde{\sigma}(\theta)))]$ is equivalent to $\mathbb{E}[u(\theta, y_1)|X \cup \tilde{X}] \geq \mathbb{E}[u(\theta, y_2)|X \cup \tilde{X}]$.*

Proof of Claim 7. Note that:

$$\begin{aligned}
\mathbb{E}[u(\theta,\mu(\sigma(\theta)))] &= \mathbb{E}[u(\theta,y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta,y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta,y_1)|X]Pr(X) \\
&\quad + \mathbb{E}[u(\theta,y_2)|\Theta_{12}\backslash X]Pr(\Theta_{12}\backslash X) + \mathbb{E}[u(\theta,y_1)|\tilde{X}]Pr(\tilde{X}) \\
&\quad + \mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash \tilde{X}]Pr(\Theta_{21}\backslash \tilde{X}). \\
\mathbb{E}[u(\theta,\tilde{\mu}(\tilde{\sigma}(\theta)))] &= \mathbb{E}[u(\theta,y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta,y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta,y_2)|X]Pr(X) \\
&\quad + \mathbb{E}[u(\theta,y_2)|\Theta_{12}\backslash X]Pr(\Theta_{12}\backslash X) + \mathbb{E}[u(\theta,y_2)|\tilde{X}]Pr(\tilde{X}) \\
&\quad + \mathbb{E}[u(\theta,y_1)|\Theta_{21}\backslash \tilde{X}]Pr(\Theta_{21}\backslash \tilde{X}).
\end{aligned}$$

Then, $\mathbb{E}[u(\theta,\mu(\sigma(\theta)))] \geq \mathbb{E}[u(\theta,\tilde{\mu}(\tilde{\sigma}(\theta)))]$ is equivalent to:

$$\begin{aligned}
& \mathbb{E}[u(\theta,y_1)|X]Pr(X) + \mathbb{E}[u(\theta,y_1)|\tilde{X}]Pr(\tilde{X}) \geq \mathbb{E}[u(\theta,y_2)|X]Pr(X) + \mathbb{E}[u(\theta,y_2)|\tilde{X}]Pr(\tilde{X}) \\
\Leftrightarrow\ & \mathbb{E}[u(\theta,y_1)|X\cup\tilde{X}]Pr(X\cup\tilde{X}) \geq \mathbb{E}[u(\theta,y_2)|X\cup\tilde{X}]Pr(X\cup\tilde{X}) \\
\Leftrightarrow\ & \mathbb{E}[u(\theta,y_1)|X\cup\tilde{X}] \geq \mathbb{E}[u(\theta,y_2)|X\cup\tilde{X}]. \qquad \square
\end{aligned}$$

Because $Pr(\Theta_{12}\backslash\bar{\Theta}_{12}) > 0$ and $Pr(X'') > 0$, by Claim 7, the hypothesis is equivalent to:

$$\mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12})\cup X''] > \mathbb{E}[u(\theta,y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12})\cup X'']. \tag{23}$$

Moreover, by the definition of $\bar{\Theta}_{12}$:

$$\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\bar{\Theta}_{12}] = \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\bar{\Theta}_{12}]. \tag{24}$$

Multiplying both sides of (23) and (24) by $Pr((\Theta_{12}\backslash\bar{\Theta}_{12})\cup X''|\Theta_{22}\cup\Theta_{12}\cup X'')$ and $Pr(\Theta_{22}\cup\bar{\Theta}_{12}|\Theta_{22}\cup\Theta_{12}\cup X'')$, respectively, and combining the results yields:

$$\mathbb{E}[u(\theta,y_1)|\Theta_{22}\cup\Theta_{12}\cup X''] > \mathbb{E}[u(\theta,y_2)|\Theta_{22}\cup\Theta_{12}\cup X'']. \tag{25}$$

However equations (22) and (25) are contradictory. Therefore, such an equilibrium $(\sigma'',\mu'';\mathcal{P}'')$ never exist. That is, any equilibrium with $X\subseteq\Theta_{21}$ attains weakly better ex ante expected utility to the receiver than equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$. By Conditions (i) and (ii), we can conclude that equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$ is one of the least informative equilibria. ■

Lemma 5 *Equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$ specified by (5) is least informative in Case (3).*

Proof of Lemma 5. This is the mirror case of Case (2), so we omit the proof. ■

Lemma 6 *Equilibrium* $(\sigma^-, \mu^-; \mathcal{P}^-)$ *specified by (6) is least informative in Case (4).*

Proof of Lemma 6. There are the following two cases to be considered:
Case (4)-1: $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})] \geq \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})]$.
Note that in equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$:

$$\mu^-(\sigma^-(\theta)) = \begin{cases} y_1 & \text{if } \theta \in \Theta_{11} \cup \Theta_{21} \cup (\Theta_{12}\backslash\bar{\Theta}_{12}) \\ y_2 & \text{if } \theta \in \Theta_{22} \cup \bar{\Theta}_{12} \\ y^R(\theta) & \text{if } \theta \in \Theta_0 \end{cases} \tag{26}$$

That is, the ex ante expected utility of the receiver in this equilibrium is equivalent to that obtained in equilibrium specified by (4). Then, because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] > \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$ holds, the same proof used in Lemma 4 is still valid in this case. Therefore, we can say that equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ is least informative in Case (4)-1.

Case 2: $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})]$.
Unlike Case (4)-1, note that in equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ in Case (4)-2:

$$\mu^-(\sigma^-(\theta)) = \begin{cases} y_1 & \text{if } \theta \in \Theta_{11} \cup \bar{\Theta}_{21} \\ y_2 & \text{if } \theta \in \Theta_{22} \cup \Theta_{12} \cup (\Theta_{21}\backslash\bar{\Theta}_{21}) \\ y^R(\theta) & \text{if } \theta \in \Theta_0 \end{cases} \tag{27}$$

That is, the ex ante expected utility of the receiver in this equilibrium is equivalent to that obtained in equilibrium specified by (5). Then, because $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ holds, the same proof used in Lemma 5 is still valid in this case. Therefore, we can say that equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ is least informative in Case (4)-2. ■

By Lemmas 3 to 6, Proposition 3 is proven. ■

Proof of Theorem 1

Because U^- and U^+ are the bounds of the receiver's ex ante expected utility, for any equilibrium, the receiver's ex ante expected utility in equilibrium must be in the interval $[U^-, U^+]$. Thus, the necessary part is obvious; it remains to prove sufficiency. Hereafter, without loss of generality, we assume that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$.

Case (1). Define the following function:

$$H(\delta) \equiv \int_{\Theta_{12}^{\delta}} (u(\theta, y_2) - u(\theta, y_1))f(\theta)d\theta + \int_{\Theta_{12}} u(\theta, y_1)f(\theta)d\theta.$$

Clearly, this function is continuous in δ. Note that by Propositions 2 and 3, $X^+ = \Theta_{12}$ and $X^- = \emptyset$. Hence:

$$\begin{aligned}
U^+ &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\qquad\qquad + \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) \\
&= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) + H(\theta_{12}^-). \\
U^- &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\qquad\qquad + \mathbb{E}[u(\theta, y_2)|\Theta_{12}]Pr(\Theta_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) \\
&= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) + H(\theta_{12}^+).
\end{aligned}$$

We fix $U \in [U^-, U^+]$ arbitrarily. From the intermediate value theorem, there exists a $\delta_U \in [\theta_{12}^-, \theta_{12}^+]$ such that:

$$\begin{aligned}
U &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) + H(\delta_U) \\
&= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) \\
&\quad + \mathbb{E}[u(\theta, y_2)|\Theta_{12}^{\delta_U}]Pr(\Theta_{12}^{\delta_U}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \backslash \Theta_{12}^{\delta_U}]Pr(\Theta_{12} \backslash \Theta_{12}^{\delta_U}).
\end{aligned}$$

We can show that there exists an equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ that supports the above partition:

$$\sigma_U(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21} \\ \Theta_{22} \cup \Theta_{12}^{\delta_U} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12}^{\delta_U} \\ \{\theta\} & \text{if } \theta \in (\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup \Theta_0 \end{cases} \tag{28}$$

Note that:

- $\mu_U(\Theta_{11} \cup \Theta_{21}) = y_1$ because of the assumption $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$;
- $\mu_U(\Theta_{22} \cup \Theta_{12}^{\delta_U}) = y_2$; $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\delta_U}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\delta_U}]$ should hold because

$\Theta_{12}^{\delta_U} \subseteq \Theta_{12}$, $u(\theta, y_1) > u(\theta, y_2)$ for all $\theta \in \Theta_{12}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$;

- $\mu_U(\{\theta\}) = y^R(\theta)$.

Given σ_U and μ_U, only types in region $\Theta_{12} \backslash \Theta_{12}^{\delta_U}$ potentially have an incentive to deviate. Then, let $S(\mathcal{P}(\cdot|m)) = m \cap (\Theta_{12} \backslash \Theta_{12}^{\delta_U})$ if $m \cap (\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \neq \emptyset$. The above off-the-equilibrium-path belief prevents those types from deviation because any message available to the types induce action y_1. Therefore, the receiver's ex ante expected utility under equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ is U.

Case (2). Note that by Proposition 3, $X^- = \Theta_{12} \backslash \bar{\Theta}_{12}$. That is:

$$\begin{aligned} U^- &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) \\ &\quad + \mathbb{E}[u(\theta, y_2)|\bar{\Theta}_{12}]Pr(\bar{\Theta}_{12}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \backslash \bar{\Theta}_{12}]Pr(\Theta_{12} \backslash \bar{\Theta}_{12}) \\ &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) + H(\bar{\delta}). \end{aligned}$$

We fix $U \in [U^-, U^+]$ arbitrarily. From the intermediate value theorem, there exists a $\delta_U \in [\theta_{12}^-, \bar{\delta}]$ such that:

$$\begin{aligned} U &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) + H(\delta_U) \\ &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_1)|\Theta_{21}]Pr(\Theta_{21}) \\ &\quad + \mathbb{E}[u(\theta, y_2)|\Theta_{12}^{\delta_U}]Pr(\Theta_{12}^{\delta_U}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \backslash \Theta_{12}^{\delta_U}]Pr(\Theta_{12} \backslash \Theta_{12}^{\delta_U}). \end{aligned}$$

The description of an equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ that supports the above partition of the state space is equivalent to that specified in Case (1). Note that $\mu_U(\Theta_{22} \cup \Theta_{12}^{\delta_U}) = y_2$; $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\delta_U}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\delta_U}]$ should hold because $\Theta_{12}^{\delta_U} \subseteq \bar{\Theta}_{12}$, $u(\theta, y_1) > u(\theta, y_2)$ for all $\theta \in \Theta_{12}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$.

Case (3). We show the statement by the following steps:

Step 1: Characterization of the least informative equilibrium.

Let $\theta_{21}^- \equiv \inf \Theta_{21}$ and $\theta_{21}^+ \equiv \sup \Theta_{21}$. Define $\Theta_{21}^{\epsilon} \equiv \{\theta \in \Theta_{21} \cup \{\theta_{21}^-, \theta_{21}^+\} | \theta_{21}^- \leq \theta \leq \epsilon\}$ for $\epsilon \in [\theta_{21}^-, \theta_{21}^+]$. Then, define:

$$J(\epsilon) \equiv \int_{\Theta_{11} \cup \Theta_{21}^{\epsilon}} (u(\theta, y_2) - u(\theta, y_1)) f(\theta) d\theta.$$

It is clear that function $J(\epsilon)$ is continuous in ϵ. Note that $J(\theta_{21}^-) = \mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11})$.

Because $u(\theta, y_1) > u(\theta, y_2)$ for any $\theta \in \Theta_{11}$ and $Pr(\Theta_{11}) > 0$, $J(\theta_{21}^{-}) < 0$. Also, note that $J(\theta_{12}^{+}) = \mathbb{E}[u(\theta, y_2) - u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}]Pr(\Theta_{11} \cup \Theta_{21})$. Because $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$ and $Pr(\Theta_{11} \cup \Theta_{21}) > 0$, $J(\theta_{21}^{+}) > 0$. Thus, by the intermediate value theorem, there exists $\bar{\epsilon} \in (\theta_{21}^{-}, \theta_{21}^{+})$ such that $J(\bar{\epsilon}) = 0$. That is, we can say that $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}^{\bar{\epsilon}}] = \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}^{\bar{\epsilon}}]$. Define $\bar{\Theta}_{21} \equiv \Theta_{21}^{\bar{\epsilon}}$. Now, define:

$$K(\epsilon) \equiv \int_{\Theta_{21}^{\epsilon}} (u(\theta, y_1) - u(\theta, y_2))f(\theta)d\theta + \int_{\Theta_{21}} u(\theta, y_2)f(\theta)d\theta.$$

Clearly, this function is continuous in ϵ. By Proposition 3:

$$\begin{aligned} U^{-} &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_2)|\Theta_{12}]Pr(\Theta_{12}) \\ &\quad + \mathbb{E}[u(\theta, y_1)|\bar{\Theta}_{21}]Pr(\bar{\Theta}_{21}) + \mathbb{E}[u(\theta, y_2)|\Theta_{21} \backslash \bar{\Theta}_{21}]Pr(\Theta_{21} \backslash \bar{\Theta}_{21}) \\ &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_2)|\Theta_{12}]Pr(\Theta_{12}) + K(\bar{\epsilon}). \end{aligned}$$

Step 2: Characterization of the threshold.

Now, we show that there exists $\hat{U} \in (U^{-}, U^{+})$ such that $\hat{U}$ is supported as ex ante expected utility of the receiver in both equilibrium $(\hat{\sigma}_{12}, \hat{\mu}_{12}; \hat{\mathcal{P}}_{12})$ with $\hat{X}_{12} \subseteq \Theta_{12}$ and equilibrium $(\hat{\sigma}_{21}, \hat{\mu}_{21}; \hat{\mathcal{P}}_{21})$ with $\hat{X}_{21} \subseteq \Theta_{21}$.

First, we construct equilibrium $(\hat{\sigma}_{21}, \hat{\mu}_{21}; \hat{\mathcal{P}}_{21})$ with $\hat{X}_{21} \subseteq \Theta_{21}$. Define:

$$\begin{aligned} \hat{U} &\equiv \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\ &\quad + \mathbb{E}[u(\theta, y_2)|\Theta_{12}]Pr(\Theta_{12}) + \mathbb{E}[u(\theta, y_2)|\Theta_{21}]Pr(\Theta_{21}). \end{aligned}$$

The construction of equilibrium $(\hat{\sigma}_{21}, \hat{\mu}_{21}; \hat{\mathcal{P}}_{21})$ is as follows:

$$\hat{\sigma}_{21}(\theta) = \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21} \cup \Theta_0 \\ \Theta_{22} \cup \Theta_{12} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12} \end{cases} \tag{29}$$

Note that:

$$\hat{\mu}_{21}(\hat{\sigma}_{21}(\theta)) = \begin{cases} y_1 & \text{if } \theta \in \Theta_{11} \\ y_2 & \text{if } \theta \in \Theta_{22} \cup \Theta_{12} \cup \Theta_{21} \\ y^R(\theta) & \text{if } \theta \in \Theta_0 \end{cases} \tag{30}$$

That is, only types in region Θ_{21} potentially have an incentive to deviate. However, these types never deviate by undertaking action y_2 as a response to any message containing elements in region Θ_{21}. Hence, $\hat{X}_{21} = \Theta_{21}$.

Next, we construct equilibrium $(\hat{\sigma}_{12}, \hat{\mu}_{12}; \hat{\mathcal{P}}_{12})$ with $\hat{X}_{12} \subseteq \Theta_{12}$. Define function $L(\delta)$ by:

$$L(\delta) \equiv \int_{(\Theta_{12}\backslash\Theta_{12}^{\delta})\cup\Theta_{21}} (u(\theta, y_1) - u(\theta, y_2))f(\theta)d\theta.$$

Note that $L(\cdot)$ is continuous in δ and $L(\theta_{21}^{+}) < 0$. In addition, because $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$ and $Pr(\Theta_{12} \cup \Theta_{21}) > 0$, $L(\theta_{21}^{-}) \geq 0$. Then, from the intermediate value theorem, there exists $\hat{\delta} \in [\theta_{21}^{-}, \theta_{21}^{+})$ such that $L(\hat{\delta}) = 0$. That is, $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}] = \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]$. The construction of equilibrium $(\hat{\sigma}_{12}, \hat{\mu}_{12}; \hat{\mathcal{P}}_{12})$ is as follows:

$$\hat{\sigma}_{12}(\theta) = \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_0 \\ (\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21} & \text{if } \theta \in (\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21} \\ \Theta_{22} \cup \Theta_{12}^{\hat{\delta}} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12}^{\hat{\delta}} \end{cases} \tag{31}$$

Note that:

- $\hat{\mu}_{12}((\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}})\cup\Theta_{21}) = y_1$ because $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}})\cup\Theta_{21}] = \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}})\cup\Theta_{21}]$;
- $\hat{\mu}_{12}(\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}) = y_2$; $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}] < \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}]$ should hold because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$ and $\Theta_{12}^{\hat{\delta}} \subset \Theta_{12}$.

That is, only types in subset $\Theta_{12}\backslash\Theta_{12}^{\delta'}$ potentially have an incentive to deviate. However, these types never deviate by undertaking action y_1 as a response to any message containing elements in region $\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}$. Hence, $\hat{X}_{12} = \Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}$. Furthermore:

$$\begin{aligned}
\mathbb{E}[u(\theta, \hat{\mu}_{12}(\hat{\sigma}_{12}(\theta)))] &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\quad + \mathbb{E}[u(\theta, y_2)|\Theta_{12}^{\hat{\delta}}]Pr(\Theta_{12}^{\hat{\delta}}) \\
&\quad + \mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]Pr((\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}) \\
&= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\quad + \mathbb{E}[u(\theta, y_2)|\Theta_{12}^{\hat{\delta}}]Pr(\Theta_{12}^{\hat{\delta}}) \\
&\quad + \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]Pr((\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}) \\
&= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) \\
&\quad + \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]Pr(\Theta_{12} \cup \Theta_{21}) \\
&= \hat{U}.
\end{aligned}$$

Step 3: Construction of an equilibrium supporting $U \in [U^-, \hat{U}]$.

Fix $U \in [U^-, \hat{U}]$ arbitrarily. Because of the continuity of function $K(\epsilon)$ in ϵ, by the intermediate value theorem, there exists $\epsilon_U \in [\theta_{21}^-, \bar{\epsilon}]$ such that:

$$\begin{aligned}
U &= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_2)|\Theta_{12}]Pr(\Theta_{12}) + K(\epsilon_U) \\
&= \mathbb{E}[u(\theta, y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta, y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta, y_2)|\Theta_{12}]Pr(\Theta_{12}) \\
&\quad + \mathbb{E}[u(\theta, y_1)|\Theta_{21}^{\epsilon_U}]Pr(\Theta_{21}^{\epsilon_U}) + \mathbb{E}[u(\theta, y_2)|\Theta_{21} \backslash \Theta_{21}^{\epsilon_U}]Pr(\Theta_{21} \backslash \Theta_{21}^{\epsilon_U}).
\end{aligned}$$

U is supported by the following equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$:

$$\sigma_U(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21}^{\epsilon_U} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21}^{\epsilon_U} \\ \Theta_{22} \cup \Theta_{12} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12} \\ \{\theta\} & \text{if } \theta \in (\Theta_{21} \backslash \Theta_{21}^{\epsilon_U}) \cup \Theta_0 \end{cases} \tag{32}$$

Note that:

- $\mu_U(\Theta_{11} \cup \Theta_{21}^{\epsilon_U}) = y_1$; $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}^{\epsilon_U}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}^{\epsilon_U}]$ holds because $\Theta_{21}^{\epsilon_U} \subseteq \bar{\Theta}_{21}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \bar{\Theta}_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \bar{\Theta}_{21}]$;
- $\mu_U(\Theta_{22} \cup \Theta_{12}) = y_2$ because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}]$.

Hence, only types in subset $\Theta_{21} \backslash \Theta_{21}^{\epsilon_U}$ potentially have an incentive to deviate. However, these types never deviate by undertaking action y_2 as a response to any message containing elements in region $\Theta_{21}\backslash\Theta_{21}^{\epsilon_U}$. Thus, equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ supports receiver's ex ante expected utility U.

Step 4: Construction of an equilibrium supporting $U \in [\hat{U}, U^+]$.

Fix $U \in [\hat{U}, U^+]$ arbitrarily. Because of the continuity of function $H(\delta)$ in δ, by the intermediate value theorem, there exists $\delta_U \in [\theta_{12}^-, \hat{\delta}]$ such that:

$$\begin{aligned} U &= \mathbb{E}[u(\theta,y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta,y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta,y_1)|\Theta_{21}]Pr(\Theta_{21}) + H(\delta_U) \\ &= \mathbb{E}[u(\theta,y_1)|\Theta_{11}]Pr(\Theta_{11}) + \mathbb{E}[u(\theta,y_2)|\Theta_{22}]Pr(\Theta_{22}) + \mathbb{E}[u(\theta,y_1)|\Theta_{21}]Pr(\Theta_{21}) \\ &\qquad + \mathbb{E}[u(\theta,y_2)|\Theta_{12}^{\delta_U}]Pr(\Theta_{12}^{\delta_U}) + \mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\Theta_{12}^{\delta_U})]Pr(\Theta_{12}\backslash\Theta_{12}^{\delta_U}). \end{aligned}$$

U is supported by the following equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$:

$$\sigma_U(\theta) = \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_0 \\ \Theta_{22} \cup \Theta_{12}^{\delta_U} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12}^{\delta_U} \\ (\Theta_{12}\backslash\Theta_{12}^{\delta_U}) \cup \Theta_{21} & \text{if } \theta \in (\Theta_{12}\backslash\Theta_{12}^{\delta_U}) \cup \Theta_{21} \end{cases} \tag{33}$$

Note that:

- $\mu_U(\Theta_{22} \cup \Theta_{12}^{\delta_U}) = y_2$; $\mathbb{E}[u(\theta,y_1)|\Theta_{22} \cup \Theta_{12}^{\delta_U}] \leq \mathbb{E}[u(\theta,y_2)|\Theta_{22} \cup \Theta_{12}^{\delta_U}]$ holds because $\Theta_{12}^{\delta_U} \subseteq \Theta_{12}$ and $\mathbb{E}[u(\theta,y_1)|\Theta_{22} \cup \Theta_{12}] \leq \mathbb{E}[u(\theta,y_2)|\Theta_{22} \cup \Theta_{12}]$;

- $\mu_U((\Theta_{12}\backslash\Theta_{12}^{\delta_U}) \cup \Theta_{21}) = y_1$; $\mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\Theta_{12}^{\delta_U}) \cup \Theta_{21}] \geq \mathbb{E}[u(\theta,y_2)|(\Theta_{12}\backslash\Theta_{12}^{\delta_U}) \cup \Theta_{21}]$ should hold because $(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \subseteq (\Theta_{12}\backslash\Theta_{12}^{\delta_U})$ and $\mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}] = \mathbb{E}[u(\theta,y_2)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]$.

Hence, only types in subset $\Theta_{12}\backslash\Theta_{12}^{\delta_U}$ potentially have an incentive to deviate. However, these types never deviate by undertaking action y_1 as a response to any message containing elements in region $\Theta_{12}\backslash\Theta_{12}^{\delta_U}$. Thus, equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ supports receiver's ex ante expected utility U.

Case (4). There are two cases to be considered.

Case (4)-1: $\mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})] \geq \mathbb{E}[u(\theta,y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})]$.

The proof in this case is same to that in Case (2) with modification to equilibrium construction. Hence, we only mention the differences. In the description of equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$, (28) should be modified as follows:

$$\sigma_U(\theta) = \begin{cases} \Theta_{11} \cup \bar{\Theta}_{21} & \text{if } \theta \in \Theta_{11} \cup \bar{\Theta}_{21} \\ \Theta_{22} \cup \Theta_{12}^{\delta_U} & \text{if } \theta \in \Theta_{22} \cup \Theta_{12}^{\delta_U} \\ (\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21}) & \text{if } \theta \in (\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21}) \\ \{\theta\} & \text{if } \theta \in \Theta_0 \end{cases} \tag{34}$$

Note that:

- $\mu_U(\Theta_{11} \cup \bar{\Theta}_{21}) = y_1$ because $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \bar{\Theta}_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \bar{\Theta}_{21}]$;
- $\mu_U(\Theta_{22} \cup \Theta_{12}^{\delta_U}) = y_2$; $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\delta_U}] \leq \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\delta_U}]$ holds because $\Theta_{12}^{\delta_U} \subseteq \bar{\Theta}_{12}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$;
- $\mu_U((\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})) = y_1$; $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] \geq \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]$ holds because $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] \geq \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]$ and $(\Theta_{12} \backslash \bar{\Theta}_{12}) \subseteq (\Theta_{12} \backslash \Theta_{12}^{\delta_U})$.

Hence, the construction of off-the-equilibrium-path beliefs is the same to that in Case (2).

Case (4)-2: $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]$**.**

The proof in this case is same to that in Case (3) with modification. Hence, we mention only the differences. In the description of equilibrium $(\hat{\sigma}_{21}, \hat{\mu}_{21}; \hat{\mathcal{P}}_{21})$, (29) should be modified as follows:

$$\hat{\sigma}_{21}(\theta) = \begin{cases} \{\theta\} & \text{if } \theta \in \Theta_{11} \cup \Theta_0 \\ \Theta_{22} \cup \bar{\Theta}_{12} & \text{if } \theta \in \Theta_{22} \cup \bar{\Theta}_{12} \\ (\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta_{21} & \text{if } \theta \in (\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta_{21} \end{cases} \tag{35}$$

Note that:

- $\hat{\mu}_{21}(\Theta_{22} \cup \bar{\Theta}_{12}) = y_2$ because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$;
- $\hat{\mu}_{21}((\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta_{21}) = y_2$; $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup \Theta_{21}]$ holds because $(\Theta_{21} \backslash \bar{\Theta}_{21}) \subset \Theta_{21}$ and $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \bar{\Theta}_{12}) \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]$.

That is, only types in region Θ_{21} potentially have an incentive to deviate. However, these types never deviate by undertaking action y_2 as a response to any message containing elements in region Θ_{21}.

In the description of equilibrium $(\hat{\sigma}_{12}, \hat{\mu}_{12}; \hat{\mathcal{P}}_{12})$, the sender's strategy (31) is unchanged, but the receiver's response is modified as follows:

- $\hat{\mu}_{12}(\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}) = y_2$ because of the following reason:
  - $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup \Theta_{21}]$ holds because $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})]$ holds;
  - $\Theta_{12}^{\hat{\delta}} \subset \bar{\Theta}_{12}$ should hold because $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup \Theta_{21}]$ and $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}] = \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]$ holds;
  - $\Theta_{12}^{\hat{\delta}} \subset \bar{\Theta}_{12}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$ imply that $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}] < \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}]$.
- $\hat{\mu}_{12}((\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}) = y_1$ because $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}] = \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]$.

Hence, the construction of off-the-equilibrium-path beliefs is the same to that in Case (3).

In the description of equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ that supports $U \in [U^-, \hat{U}]$, (32) is modified as follows:

$$\sigma_U(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21}^{\epsilon_U} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21}^{\epsilon_U} \\ \Theta_{22} \cup \bar{\Theta}_{12} & \text{if } \theta \in \Theta_{22} \cup \bar{\Theta}_{12} \\ (\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\Theta_{21}^{\epsilon_U}) & \text{if } \theta \in (\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\Theta_{21}^{\epsilon_U}) \\ \{\theta\} & \text{if } \theta \in \Theta_0 \end{cases} \tag{36}$$

Note that:

- $\mu_U(\Theta_{11} \cup \Theta_{21}^{\epsilon_U}) = y_1$; $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}^{\epsilon_U}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}^{\epsilon_U}]$ holds because $\Theta_{21}^{\epsilon_U} \subseteq \bar{\Theta}_{21}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \bar{\Theta}_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \bar{\Theta}_{21}]$;[18]
- $\mu_U(\Theta_{22} \cup \bar{\Theta}_{12}) = y_2$ because $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \bar{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \bar{\Theta}_{12}]$;
- $\mu_U((\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\Theta_{21}^{\epsilon_U})) = y_2$; $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\Theta_{21}^{\epsilon_U})] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\Theta_{21}^{\epsilon_U})]$ holds because $(\Theta_{21}\backslash\bar{\Theta}_{21}) \subseteq (\Theta_{21}\backslash\Theta_{21}^{\epsilon_U})$ and $\mathbb{E}[u(\theta, y_1)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})] < \mathbb{E}[u(\theta, y_2)|(\Theta_{12}\backslash\bar{\Theta}_{12}) \cup (\Theta_{21}\backslash\bar{\Theta}_{21})]$.

Hence, the construction of off-the-equilibrium-path beliefs is the same to that in Case (3).

In the description of equilibrium $(\sigma_U, \mu_U; \mathcal{P}_U)$ that supports $U \in [\hat{U}, U^+]$, the sender's strategy (33) is unchanged, but the receiver's response is modified as follows:

[18]Note that $\epsilon_U \leq \bar{\epsilon}$ implies that $\Theta_{21}^{\epsilon_U} \subseteq \Theta_{21}^{\bar{\epsilon}} = \bar{\Theta}_{21}$.

- $\mu_U(\Theta_{22} \cup \Theta_{12}^{\delta_U}) = y_2$; $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\delta_U}] < \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\delta_U}]$ should hold because $\Theta_{12}^{\delta_U} \subseteq \Theta_{12}^{\hat{\delta}}$ and $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}] < \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \Theta_{12}^{\hat{\delta}}]$;[19]

- $\mu_U((\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup \Theta_{21}) = y_1$; $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \Theta_{12}^{\delta_U}) \cup \Theta_{21}]$ should hold because $(\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \subseteq (\Theta_{12} \backslash \Theta_{12}^{\delta_U})$ and $\mathbb{E}[u(\theta, y_1)|(\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}] = \mathbb{E}[u(\theta, y_2)|(\Theta_{12} \backslash \Theta_{12}^{\hat{\delta}}) \cup \Theta_{21}]$.

Hence, the construction of off-the-equilibrium-path beliefs is the same to that in Case (3). Therefore, the sufficiency is proven. ■

Proof of Corollary 1

Suppose that there exists the full disclosure equilibrium. It is easily shown that the necessary and sufficient condition for the existence of the full disclosure equilibrium is either region $\Theta_{12} = \emptyset$ or region $\Theta_{21} = \emptyset$. Hence, without loss of generality, we assume that $\Theta_{21} = \emptyset$. Note that because the proof of Theorem 1 does not depend on the property that $Pr(\Theta_{21}) > 0$, it is sufficient to show that the most and the least informative equilibria in this environment are also characterized by the propositions in the body of the paper. It is obvious that the equilibrium characterized in Proposition 2 is one of the full disclosure equilibria; that is, the most informative equilibrium is specified by Proposition 2. The characterization of the least informative equilibrium in Proposition 3 is also not affected by this modification.

Next, suppose that there exists the full pooling equilibrium denoted by $(\sigma_P, \mu_P; \mathcal{P}_P)$. By Lemma 1, either region $\Theta_{11} = \emptyset$ or region $\Theta_{22} = \emptyset$ should hold. Without loss of generality, assume that region $\Theta_{22} = \emptyset$. Note that it is also necessary that $\mu_P(\Theta) = y_1$; otherwise, types in agreement region Θ_{11} deviate to disclosure messages. Again, because the proof of Theorem 1 does not depend on the property that $Pr.(\Theta_{22}) > 0$, it is sufficient to show that the most and the least informative equilibria are characterized by the propositions in the body of the paper. Note that the most informative equilibrium is characterized by Proposition 2 because that proof does not depend on the property that $Pr(\Theta_{22}) > 0$.

In this environment, the least informative equilibrium is the full pooling equilibrium, and the informativeness of this equilibrium is equivalent to that specified in the propositions. There are the following two cases to be considered. Note that $\mathbb{E}[u(\theta, \mu_P(\sigma_P(\theta)))] = \mathbb{E}[u(\theta, y_1)]$.

Case 1: $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$**.**

[19]Note that $\delta_U \leq \hat{\delta}$ implies that $\Theta_{12}^{\delta_U} \subseteq \Theta_{12}^{\hat{\delta}}$.

Consider equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ specified by (4). Because $\Theta_{22} = \emptyset$, $\bar{\Theta}_{12} = \emptyset$. Hence, the ex ante expected utility in equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ is:

$$\begin{aligned} U^- &= \mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}]Pr(\Theta_{11} \cup \Theta_{21}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12}]Pr(\Theta_{12}) \\ &= \mathbb{E}[u(\theta, \mu_P(\sigma_P(\theta)))]. \end{aligned}$$

That is, equilibria $(\sigma_P, \mu_P; \mathcal{P}_P)$ and $(\sigma^-, \mu^-; \mathcal{P}^-)$ have the same informativeness.

Case 2: $\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \Theta_{21}]$.

Consider equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ specified by (6). Again, $\bar{\Theta}_{12} = \emptyset$ because $\Theta_{22} = \emptyset$.

Claim 8 $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] \geq \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]$.

Proof of Claim 8. Suppose, in contrast, that:

$$\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})] < \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]. \tag{37}$$

By definition of region $\bar{\Theta}_{21}$:

$$\mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \bar{\Theta}_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{11} \cup \bar{\Theta}_{21}]. \tag{38}$$

Multiplying the both sides of (37) and (38) by $Pr(\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21}))$ and $Pr(\Theta_{11} \cup \bar{\Theta}_{21})$, respectively, and combining the results implies that $\mathbb{E}[u(\theta, y_1)] < \mathbb{E}[u(\theta, y_2)]$, which is a contradiction to that there exists the full pooling equilibrium. □

By Claim 8, the ex ante expected utility in equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ is as follows:

$$\begin{aligned} U^- &= \mathbb{E}[u(\theta, y_1)|\Theta_{11} \cup \bar{\Theta}_{21}]Pr(\Theta_{11} \cup \bar{\Theta}_{21}) + \mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})]Pr(\Theta_{12} \cup (\Theta_{21} \backslash \bar{\Theta}_{21})) \\ &= \mathbb{E}[u(\theta, \mu_P(\sigma_P(\theta)))]. \end{aligned}$$

That is, equilibria $(\sigma_P, \mu_P; \mathcal{P}_P)$ and $(\sigma^-, \mu^-; \mathcal{P}^-)$ have the same informativeness. Therefore, the least informative equilibrium is characterized by Proposition 3. ■

Proof of Corollary 2

The model is modified as follows. Let $\sigma : \Theta \to \Delta(M)$ be the sender's strategy, and $\mu : M \to \Delta(Y)$ be the receiver's strategy. Especially, $\mu(m)$ represents the probability that the receiver undertakes action y_1 when she observes message $m \in M$. Except for this modification, the model is identical to that defined in Section 2. Let $R \equiv \{\theta \in \Theta | \mu(\sigma(\theta)) \in (0,1)\}$ be the set of types where the sender induces the receiver's completely mixed response in equilibrium $(\sigma, \mu; \mathcal{P})$. The outline of the proof is as mentioned in the body of the paper. First, we show useful lemmas.

Lemma 7 *For any equilibrium $(\sigma, \mu; \mathcal{P})$ with $R \neq \emptyset$, $\mathbb{E}[u(\theta, y_1)|R] = \mathbb{E}[u(\theta, y_2)|R]$ holds.*

Proof of Lemma 7. Let $M_R \equiv \{m \in M | \mu(m) \in (0,1)$ and there exists $\theta \in R$ s.t. $m \in \mathcal{S}(\sigma(\theta))\}$ be the set of on-the-equilibrium-path messages that induce the receiver's completely mixed response. Note that $\mathbb{E}_{\mathcal{P}(\cdot|m)}[u(\theta, y_1)] = \mathbb{E}_{\mathcal{P}(\cdot|m)}[u(\theta, y_2)]$ for any $m \in M_R$ because the receiver randomizes given message m. Without loss of generality, assume that set M_R is countable. Then:

$$\begin{aligned}
\mathbb{E}[u(\theta, y_1)|R] &= \sum_{m \in M_R} Pr(m) \mathbb{E}_{\mathcal{P}(\cdot|m)}[u(\theta, y_1)] \\
&= \sum_{m \in M_R} Pr(m) \mathbb{E}_{\mathcal{P}(\cdot|m)}[u(\theta, y_2)] \\
&= \mathbb{E}[u(\theta, y_2)|R].
\end{aligned}$$

For the scenario where M_R is uncountable can be shown by the similar argument. Therefore, $\mathbb{E}[u(\theta, y_1)|R] = \mathbb{E}[u(\theta, y_2)|R]$ holds. ■

Lemma 8 *Suppose that Assumption 1 holds. Then, for any equilibrium $(\sigma, \mu; \mathcal{P})$ with $R \neq \emptyset$, (i) $(\Theta_{11} \cup \Theta_{22}) \cap R = \emptyset$, and (ii) $Pr(R) < Pr(\Theta_{12} \cup \Theta_{21})$.*

Proof of Lemma 8. (i) Suppose, in contrast, that there exists an equilibrium $(\sigma, \mu; \mathcal{P})$ with $R \neq \emptyset$ such that $(\Theta_{11} \cup \Theta_{22}) \cap R \neq \emptyset$. However, by Lemma 1, type $\theta \in (\Theta_{11} \cup \Theta_{22}) \cap R$ deviates to message $\{\theta\}$, which is a contradiction.

(ii) Suppose, in contrast, that there exists an equilibrium $(\sigma, \mu; \mathcal{P})$ with $R \neq \emptyset$ such that $Pr(R) \geq Pr(\Theta_{12} \cup \Theta_{21})$. By the first half of this lemma, $R \subseteq \Theta_{12} \cup \Theta_{21} \cup \Theta_0$. That is, $Pr(R) = Pr(\Theta_{12} \cup \Theta_{21})$ should hold. That is, almost every type in region Θ_{12} is pooling with some types in region Θ_{21} with inducing receiver's completely mixed response, and so is almost every type in region Θ_{21}. By Lemma 7, $\mathbb{E}[u(\theta, y_1)|R] = \mathbb{E}[u(\theta, y_2)|R]$ must hold. However, because $Pr(R) = Pr(\Theta_{12} \cup \Theta_{21})$, $\mathbb{E}[u(\theta, y_1)|R] = \mathbb{E}[u(\theta, y_2)|R]$ implies that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] = \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$, which is a contradiction to Assumption 1-(i). ■

Proposition 4 *Suppose that Assumption 1 holds. Then, for any equilibrium* $(\sigma, \mu; \mathcal{P})$ *with* $R \neq \emptyset$, $\mathbb{E}[u(\theta, \mu^+(\sigma^+(\theta)))] \geq \mathbb{E}[u(\theta, \mu(\sigma(\theta)))]$ *holds.*

Proof of Proposition 4. Without less of generality, assume that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] > \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$. Suppose, in contrast, that there exists an equilibrium $(\sigma, \mu; \mathcal{P})$ with $R \neq \emptyset$ such that:

$$\mathbb{E}[u(\theta, \mu^+(\sigma^+(\theta)))] < \mathbb{E}[u(\theta, \mu(\sigma(\theta)))]. \tag{39}$$

By Lemmas 1, 7, 8, and Proposition 2, (39) is equivalent to:

$$\mathbb{E}[u(\theta, \mu(\sigma(\theta)))|(\Theta_{12} \cup \Theta_{21}) \cap R^c] > \mathbb{E}[u(\theta, y_1)|(\Theta_{12} \cup \Theta_{21}) \cap R^c], \tag{40}$$

where $R^c \equiv \Theta \backslash R$ is the set of types where $\mu(\sigma(\theta))$ is deterministic.

Claim 9 *Both* $\Theta_{12} \cap R^c \neq \emptyset$ *and* $\Theta_{21} \cap R^c \neq \emptyset$ *hold.*

Proof of Claim 9. Suppose, in contrast, that either $\Theta_{12} \cap R^c = \emptyset$ or $\Theta_{21} \cap R^c = \emptyset$. Suppose that $\Theta_{12} \cap R^c = \emptyset$, and then $\Theta_{12} \subset R$. Hence, by the similar argument in Lemma 7, it should hold that:

$$\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup (\Theta_{21} \cap R)] = \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup (\Theta_{21} \cap R)]. \tag{41}$$

By Lemma 8, $Pr((\Theta_{21} \cap R)) < Pr(\Theta_{21})$. Then, because $u(\theta, y_1) < y(\theta, y_2)$ for any $\theta \in \Theta_{21}$, (41) implies that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] < \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$, which is a contradiction to the assumption that $\mathbb{E}[u(\theta, y_1)|\Theta_{12} \cup \Theta_{21}] > \mathbb{E}[u(\theta, y_2)|\Theta_{12} \cup \Theta_{21}]$.

Next, suppose that $\Theta_{21} \cap R^c = \emptyset$. That is, $(\Theta_{12} \cup \Theta_{21}) \cap R^c \subseteq \Theta_{12}$. However, because $u(\theta, y_1) > u(\theta, y_2)$ for any $\theta \in (\Theta_{12} \cup \Theta_{21}) \cap R^c$, $\mathbb{E}[u(\theta, \mu(\sigma(\theta)))|(\Theta_{12} \cup \Theta_{21}) \cap R^c] \leq \mathbb{E}[u(\theta, y_1)|(\Theta_{12} \cup \Theta_{21}) \cap R^c]$ should hold, which is a contradiction to (40). Therefore, both $\Theta_{12} \cap R^c \neq \emptyset$ and $\Theta_{21} \cap R^c \neq \emptyset$ hold. □

Because of Claim 9, we can derive a contradiction by the similar argument in the proof of Proposition 2 starting from Claim 1 where regions Θ_{12} and Θ_{21} are replaced by regions $\Theta_{12} \cap R^c$ and $\Theta_{21} \cap R^c$, respectively.[20] Therefore, $\mathbb{E}[u(\theta, \mu^+(\sigma^+(\theta)))] \geq \mathbb{E}[u(\theta, \mu(\sigma(\theta)))]$ holds. ■

[20]The complete proof is in the Supplementary Appendix.

Proposition 5 *Suppose that Assumption 1 holds. Then, for any equilibrium* $(\sigma, \mu; \mathcal{P})$ *with* $R \neq \emptyset$*,* $\mathbb{E}[u(\theta, \mu^-(\sigma^-(\theta)))] \leq \mathbb{E}[u(\theta, \mu(\sigma(\theta)))]$ *holds.*

Proof of Proposition 5. Suppose, in contrast, that there exists equilibrium $(\sigma, \mu; \mathcal{P})$ with $R \neq \emptyset$ such that:

$$\mathbb{E}[u(\theta, \mu^-(\sigma^-(\theta)))] > \mathbb{E}[u(\theta, \mu(\sigma(\theta)))]. \tag{42}$$

Because of Assumption 1, it is sufficient to consider Cases (1) to (3).

Case (1). It is obvious that equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$ attains the minimum even if mixed strategies are allowed because the first-best action $y^R(\theta)$ is never induced over the disagreement region in equilibrium $(\sigma^-, \mu^-; \mathcal{P}^-)$, which is a contradiction to (42).

Case (2). We consider the following new equilibrium $(\tilde{\sigma}, \tilde{\mu}; \tilde{\mathcal{P}})$ such that:

$$\tilde{\sigma}(\theta) = \begin{cases} \Theta_{11} \cup \Theta_{21} & \text{if } \theta \in \Theta_{11} \cup \Theta_{21} \\ \Theta_{22} \cup \tilde{\Theta}_{12} & \text{if } \theta \in \Theta_{22} \cup \tilde{\Theta}_{12} \\ \{\theta\} & \text{if } \theta \in (\Theta_{12} \backslash \tilde{\Theta}_{12}) \cup \Theta_0, \end{cases} \tag{43}$$

where $\tilde{\Theta}_{12}$ is a subset of disagreement region Θ_{12} such that (i) $\mathbb{E}[u(\theta, y_1)|\Theta_{22} \cup \tilde{\Theta}_{12}] = \mathbb{E}[u(\theta, y_2)|\Theta_{22} \cup \tilde{\Theta}_{12}]$, and (ii) if $\tilde{\Theta}_{12} \cap R \neq \emptyset$, then $(\Theta_{12} \cap R^c) \subseteq \tilde{\Theta}_{12}$. That is, by Claim 6, $\mathbb{E}[u(\theta, \tilde{\mu}(\tilde{\sigma}(\theta)))] = \mathbb{E}[u(\theta, \mu^-(\sigma^-(\theta)))] = U^-$, and region $\tilde{\Theta}_{12}$ is constructed by types in region $\Theta_{12} \cap R^c$ as much as possible. Then, we compare equilibria $(\tilde{\sigma}, \tilde{\mu}; \tilde{\mathcal{P}})$ and $(\sigma, \mu; \mathcal{P})$. That is, (42) is equivalent to:

$$\begin{aligned} &\mathbb{E}[u(\theta, \tilde{\mu}(\tilde{\sigma}(\theta)))|R^c]Pr(R^c) + \mathbb{E}[u(\theta, \tilde{\mu}(\tilde{\sigma}(\theta)))|R]Pr.(R) \\ &> \quad \mathbb{E}[u(\theta, \mu(\sigma(\theta)))|R^c]Pr(R^c) + \mathbb{E}[u(\theta, \mu(\sigma(\theta)))|R]Pr.(R). \end{aligned} \tag{44}$$

First, compare these equilibria over region R^c where the receiver's response is deterministic in equilibrium $(\sigma, \mu; \mathcal{P})$. Note that equilibrium $(\tilde{\sigma}, \tilde{\mu}; \tilde{\mathcal{P}})$ attains the minimum ex ante expected utility conditional on region R^c because of Proposition 3 and the construction of equilibrium $(\tilde{\sigma}, \tilde{\mu}; \tilde{\mathcal{P}})$:

$$\mathbb{E}[u(\theta, \tilde{\mu}(\tilde{\sigma}(\theta)))|R^c] \leq \mathbb{E}[u(\theta, \mu(\sigma(\theta)))|R^c].$$

Hence, to hold (44), it is necessary that:

$$\mathbb{E}[u(\theta,\tilde{\mu}(\tilde{\sigma}(\theta)))|R]Pr(R) > \mathbb{E}[u(\theta,\mu(\sigma(\theta)))|R]Pr(R). \tag{45}$$

Note that:

$$\begin{aligned}\mathbb{E}[u(\theta,\tilde{\mu}(\tilde{\sigma}(\theta)))|R]Pr(R) &= \mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\tilde{\Theta}_{12})\cap R]Pr((\Theta_{12}\backslash\tilde{\Theta}_{12})\cap R)\\ &\quad +\mathbb{E}[u(\theta,y_2)|\tilde{\Theta}_{12}\cap R]Pr(\tilde{\Theta}_{12}\cap R)\\ &\quad +\mathbb{E}[u(\theta,y_1)|\Theta_{21}\cap R]Pr(\Theta_{21}\cap R).\end{aligned} \tag{46}$$

For any $m\in M_R$, $\mathbb{E}_{\mathcal{P}(\cdot|m)}[u(\theta,\mu(\sigma(\theta)))] = \mathbb{E}_{\mathcal{P}(\cdot|m)}[u(\theta,y_1)]$. Hence, by the similar argument in Lemma 7, we can say that $\mathbb{E}[u(\theta,\mu(\sigma(\theta)))|R] = \mathbb{E}[u(\theta,y_1)|R]$. That is:

$$\begin{aligned}\mathbb{E}[u(\theta,\mu(\sigma(\theta)))|R]Pr(R) &= \mathbb{E}[u(\theta,y_1)|(\Theta_{12}\backslash\tilde{\Theta}_{12})\cap R]Pr((\Theta_{12}\backslash\tilde{\Theta}_{12})\cap R)\\ &\quad +\mathbb{E}[u(\theta,y_1)|\tilde{\Theta}_{12}\cap R]Pr(\tilde{\Theta}_{12}\cap R)\\ &\quad +\mathbb{E}[u(\theta,y_1)|\Theta_{21}\cap R]Pr(\Theta_{21}\cap R).\end{aligned} \tag{47}$$

Then, (46) and (47) implies that (45) is equivalent to:

$$\mathbb{E}[u(\theta,y_2)|\tilde{\Theta}_{12}\cap R]Pr(\tilde{\Theta}_{12}\cap R) > \mathbb{E}[u(\theta,y_1)|\tilde{\Theta}_{12}\cap R]Pr(\tilde{\Theta}_{12}\cap R). \tag{48}$$

However, because $u(\theta,y_1) > u(\theta,y_2)$ for any $\theta\in\tilde{\Theta}_{12}\cap R$, (48) is impossible, which is a contradiction.

Case (3). We can derive a contradiction by the analogy of Case (2).

Therefore, there exists no equilibrium $(\sigma,\mu;\mathcal{P})$ with $R\neq\emptyset$ being dominated by equilibrium $(\sigma^-,\mu^-;\mathcal{P}^-)$. ■

Proof of Corollary 2. By Propositions 4 and 5, without loss of generality, we can restrict our attention to the ceases where the receiver adopts pure strategies. Furthermore, as long as the receiver adopts pure strategies, the proofs of Propositions 2 and 3 do not depend on whether the sender adopts pure strategies. In other words, the characterizations of the most and the least informative equilibria do not change even if the players adopt mixed strategies. Again, because the proof of

Theorem 1 does not depend on whether the players adopt pure strategies, the theorem is still valid even if mixed strategies are allowed. ■

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