On the possibility of information transmission: a (costly) signaling case

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1 Introduction

In our main article, Miura and Yamashita (2014) ("On the possibility of information transmission"), we have shown in a cheap-talk environment that "assuming full revelation in a common-knowledge environment" implies a very different conclusion for models with slight misspecification in the sense of the topology of convergence in probability.

In this note, we obtain a similar no-prediction result in another example. The example is about a costly signaling environment (Spence (1973)). That is, the main difference from the cheap-talk case is that the sender's message is costly.

2 Model

A sender is a job-market candidate, whose ability $\theta \in \Theta = \mathbb{R}_{++}$ is his private information. He chooses an effort level $e \in E = \mathbb{R}_+$, as a (costly) message

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for his ability, which costs $\frac{e^2}{2\theta}$ to him.[1] The sender's utility is $w + de - \frac{e^2}{2\theta}$, where $w \in W = \mathbb{R}_+$ is the wage paid by the employer, and $d \in D_\varepsilon = [-\varepsilon, \varepsilon]$ represents an additional incentive for effort (e.g., human capital not directly relevant to production). In the benchmark case, $d = 0$ is common knowledge.

A receiver is a potential employer who offers the wage $w(e) \in \mathbb{R}_+$ after she observes the effort level e chosen by the sender. We assume $w(0) = 0$ for normalization, and this essentially serves as the sender's outside option. For $e > 0$, the receiver offers the competitive wage level, i.e., $w(e)$ equals the expected value of θ given the observed effort choice e.[2] The prior over θ is assumed to be common knowledge throughout the analysis.

3 A benchmark case: $d = 0$ being common knowledge

First, we consider a benchmark case where $d = 0$ is common knowledge among the players. As in the standard approach in the literature, we assume that the players play a fully-revealing perfect-Bayesian equilibrium, i.e., the sender's effort in state θ is given by $e(\theta)$ and the receiver's wage offer given each e is given by $w(e)$ such that e and w are injective. By the result of Mailath and von Thadden (2013) (Theorem 2.1), the fully-revealing strategy $e(\cdot)$ should be differentiable on Θ.

In the next section, we study the implication of this assumption on the equilibrium behaviors when the model is slightly misspecified in the sense of the topology of convergence in probability, as in Miura and Yamashita (2014).

[1] Following the convention, the sender is treated as male and the receiver as female throughout the paper.

[2] We can interpret that this receiver is one of the potential employers who are symmetric both in the sense of the preferences and beliefs. For example, each potential employer can earn a net profit of θ by employing the sender, and hence, through a Bertrand competition among them, the wage equals the expected value of θ conditional on the information available.

Lemma 1. A fully-revealing equilibrium exists and unique in the benchmark environment. More specifically, the sender's effort in each state $\theta \in \Theta$ is given by $e(\theta) = \theta$ and the receiver's wage offer given each $e \in E$ is given by $w(e) = e$.

Proof. We first show that the strategy profile in the statement comprises a perfect Bayesian equilibrium. Because $w(e) = E(\theta|e)$, it suffices to show that $e(\theta) = \theta$ maximizes the sender's expected utility given the wage function.

Indeed, the sender's utility in state θ is

$$e - \frac{e^2}{2\theta},$$

which is maximized at $e = \theta$.

We now show its uniqueness within the class of fully-revealing equilibria. First, because $w(e) - \frac{e^2}{2\theta}$ has an increasing difference in (e, θ), $e(\theta)$ is non-decreasing. Therefore, full revelation necessarily implies that $e(\theta)$ is strictly increasing.

Because the set of effort levels, E, is an open set, the optimal effort choice given any θ satisfies the first-order condition: $w'(e) = \frac{e}{\theta}$. Let $\theta^*(e) = \frac{e}{w'(e)}$ be the inverse function of the equilibrium choice of effort. Then the wage satisfies that $w(e) = E(\theta|e) = \theta^*(e) = \frac{e}{w'(e)}$ (for any e in the interior of the interval $X = \{e(\theta)|\theta > 0\}$).

The solution to this differential equation is $w(e)^2 - e^2 = c$ for some constant c, or $w(e) = \sqrt{e^2 + c}$, for $e \in \text{int}X$. Given such a wage schedule, the first-order condition of the sender's payoff gives $\theta^2 = e^2 + c$, or $e(\theta) = \sqrt{\theta^2 - c}$. For this condition to make sense for any $\theta > 0$, we must have $c \leq 0$.

Note that the sender's payoff cannot be negative in any θ; otherwise, he can profitably deviate to $e = 0$. For $\theta - \frac{\theta^2 - c}{2\theta} = \frac{\theta}{2} + \frac{c}{2\theta}$ to be nonnegative, we must have $c \geq 0$. Therefore, we obtain $c = 0$, which implies that $w(e)^2 = e^2$ or $w(e) = e$, and furthermore, $e(\theta) = \theta$. $\square$

4 When d is close to zero in the topology of convergence in probability

We now consider a situation where $d \in [-\varepsilon, \varepsilon]$ is common knowledge, and obtain a qualitatively similar no-prediction result as in Miura and Yamashita (2014). Specifically, we consider the same Harsanyi's type space $\mathcal{T} = (T_1, T_2, b_1, b_2)$ as in Miura and Yamashita (2014). We describe $\mathcal{T}$ for the completeness. In the following, for each type of the sender, $t_1 \in T_1$, we denote by $d(t_1) \in D_\varepsilon$ and $\theta(t_1) \in \Theta$ denote what t_1 knows about the parameter d and the state θ, and we denote by $b_i : T_i \to \Delta(T_{-i})$ the belief of player i about the other player's type.

First, define T_i^0 for each i as follows. Let $T_1^0 = \{t_1^0(\theta) | \theta \in \Theta\}$ be a subset of types of the sender which we refer to as "level-0" types, where for each θ, $t_1^0(\theta)$ is a type of the sender who (i) has $d = 0$, (ii) knows the state θ, and (iii) believes that the receiver's type is in T_2^0, i.e.,

$$d(t_1^0(\theta)) = 0, \ \theta(t_1^0(\theta)) = \theta, \text{ and } b_1(T_2^0 | t_1^0(\theta)) = 1.$$

Let $T_2^0 = \{t_2^0\}$, where t_2^0 is a "level-0" type of the receiver who believes that the sender's type is in T_1^0 (i.e., $b_2(T_1^0 | t_2^0) = 1$).

Note that $d = 0$ is commonly believed among them. Therefore, we will assume a fully-revealing and differentiable equilibrium plays for them.

Next, for each $d \in D_\varepsilon$, let $T_1^1(d) = \{t_1^1(d, \theta) | \theta \in \Theta\}$ be another subset of types of the sender ("level-1" types), where for each θ, $t_1^1(d, \theta)$ is a type of the sender who (i) has d, (ii) knows the state θ, and (iii) believes that the receiver's type is t_2^0 for certain, i.e.,

$$d(t_1^1(d, \theta)) = d, \ \theta(t_1^1(d, \theta)) = \theta, \text{ and } b_1(T_2^0 | t_1^1(\theta)) = 1.$$

Let $T_1^1 = \bigcup_{d \in D_\varepsilon} T_1^1(d)$.

Let $T_2^1 = \{t_2^1(d) | d \in D_\varepsilon\}$ be a set of "level-1" types of the receiver, where for each $d \in D_\varepsilon$, $t_2^1(d)$ believes that the sender's type is in $T_1^1(d)$ (i.e., $b_2(T_1^1(d) | t_2^1(d)) = 1$).

Inductively, given T_2^k for each $k = 1, 2, \ldots$, let T_1^{k+1} be another subset of the sender's types ("level-$(k+1)$" types)as follows. First, for each $d \in D_\varepsilon$ and $t_2 \in T_2^k$, let $T_1^{k+1}(d, t_2)$ be such that $T_1^{k+1}(d, t_2) = \{t_1^{k+1}(d, \theta, t_2) | \theta \in \Theta\}$, where for each θ, $t_1^{k+1}(d, \theta, t_2)$ is a type of the sender who (i) has the bias d, (ii) knows the state θ, and (iii) believes that the receiver's type is t_2 for certain, i.e.,

$$d(t_1^{k+1}(d, \theta, t_2)) = d, \ \theta(t_1^{k+1}(d, \theta, t_2)) = \theta, \text{ and } b_1(T_2^k | t_1^{k+1}(\theta)) = 1.$$

Let $T_1^{k+1} = \bigcup_{d \in D_\varepsilon, t_2 \in T_2^k} T_1^{k+1}(d, t_2)$.

Similarly, let T_2^{k+1} be another subset of the receiver's types ("level-$(k+1)$" types) as follows. We let $T_2^{k+1} = \{t_2^{k+1}(d, t_2) | d \in D_\varepsilon, t_2 \in T_2^k\}$, where, for each $d \in D_\varepsilon$ and $t_2 \in T_2^k$, $t_2^{k+1}(d, t_2)$ believes that the sender's type is in $T_1^{k+1}(d, t_2)$ (i.e., $b_2(T_1^{k+1}(d, t_2) | t_2^{k+1}(d, t_2)) = 1$).

We complete the description of the type space by defining $T_i = \bigcup_{k=0}^\infty T_i^k$ for each i. One interpretation may be that type 0 is the "naive" type who believes that there is no conflict in their preferences. A type of the sender in T_1^k tries to best respond to a type of the receiver in T_2^{k-1}, and a type of the receiver in T_2^k tries to best respond to a type of the sender in T_1^k.

Let $\sigma_1 : T_1 \to E$ denote the sender's (pure) effort choice, and $\sigma_2 : T_2 \times E \to W$ denote the receiver's (pure) wage offer. Let $\sigma^* = (\sigma_1^*, \sigma_2^*)$ denote a perfect Bayesian equilibrium in the game. Let $E^*(\theta), W^*(\theta)$ denote the set of equilibrium efforts and wages that can occur in state θ, i.e.,

$$\begin{aligned} E^*(\theta) &= \left\{ \sigma_1^*(t_1) \;\middle|\; t_1 \in T_1 \text{ s.t. } d(t_1) \in D_\varepsilon, \ \theta(t_1) = \theta \right\}. \\ W^*(\theta) &= \left\{ \sigma_2^*\big(t_2 | \sigma_1^*(t_1)\big) \;\middle|\; (t_1, t_2) \in T \text{ s.t. } d(t_1) \in D_\varepsilon, \ \theta(t_1) = \theta \right\}. \end{aligned}$$

We say that a perfect Bayesian equilibrium given type space $\mathcal{T}$ satisfies *Property FRD0* if, in any belief-closed subset where $d = 0$ is commonly believed, the sender's effort is θ given any θ and the receiver's wage offer is e given any e.

Theorem 1. In any equilibrium with Property FRD0, for any $\theta \in \Theta$, we have $E^*(\theta) = E$ and $W^*(\theta) = W$.

Proof. Let $e_0(\theta) = \theta$ and $w_0(e) = e$.

First, consider the sender with a "level-1" type who knows d. His payoff is

$$e + de - \frac{e^2}{2\theta}.$$

Because he believes that the receiver has the "level-0" type, he chooses

$$e_1(\theta) = (1+d)\theta \in [(1-\varepsilon)\theta, (1+\varepsilon)\theta] = [\underline{z}_1\theta, \overline{z}_1\theta],$$

where $\underline{z}_1 = 1-\varepsilon$ and $\overline{z}_1 = 1+\varepsilon$.

Believing that the sender is one of such "level-1" types, the receiver offers the wage

$$w_1(e) = \frac{e}{1+d} \in [\frac{e}{\overline{z}_1}, \frac{e}{\underline{z}_1}].$$

Now, for the sender who knows d and believes that the receiver is one of those "level-1" types, his payoff is

$$\frac{e}{1+d'} + de - \frac{e^2}{2\theta}.$$

Thus, he chooses

$$e_2(\theta) = (\frac{1}{1+d'} + d)\theta \in [(\frac{1}{1+\varepsilon} - \varepsilon)\theta, (\frac{1}{1-\varepsilon} + \varepsilon)\theta] = [\underline{z}_2\theta, \overline{z}_2\theta],$$

where $\underline{z}_2 = \frac{1}{1+\varepsilon} - \varepsilon$ and $\overline{z}_2 = \frac{1}{1-\varepsilon} + \varepsilon$.

Believing that the sender is one of such "level-2" types, the receiver offers the wage

$$w_2(e) = \frac{e}{\frac{1}{1+d'} + d} \in [\frac{e}{\frac{1}{1-\varepsilon} + \varepsilon}, \frac{e}{\frac{1}{1+\varepsilon} - \varepsilon}] = [\frac{e}{\overline{z}_2}, \frac{e}{\underline{z}_2}].$$

By induction, suppose that, for some $0 < \underline{z}_k < \overline{z}_k$, we have

$$e_k(\theta) \in [\underline{z}_k\theta, \overline{z}_k\theta],$$

and

$$w_k(e) \in [\frac{e}{\overline{z}_k}, \frac{e}{\underline{z}_k}].$$

Then, for the sender who knows d and believes that the receiver is one of the "level-k" types, his payoff is

$$\frac{e}{z_k} + de - \frac{e^2}{2\theta},$$

for some $z_k \in [\underline{z}_k, \overline{z}_k]$. Thus, he chooses

$$e_{k+1}(\theta) = (\frac{1}{z_k} + d)\theta \in [\underline{z}_{k+1}\theta, \overline{z}_{k+1}\theta],$$

where $\underline{z}_{k+1} = \frac{1}{\overline{z}_k} - \varepsilon$ and $\overline{z}_{k+1} = \frac{1}{\underline{z}_k} + \varepsilon$.

Believing that the sender is one of such "level-$(k+1)$" types, the receiver offers the wage

$$w_{k+1}(e) \in [\frac{e}{\overline{z}_{k+1}}, \frac{e}{\underline{z}_{k+1}}].$$

We can continue this argument unless $\underline{z}_{k+1} \leq 0$ or $\overline{z}_{k+1} \leq 0$. Let k^* be the first integer such that $\underline{z}_{k^*} \leq 0$ or $\overline{z}_{k^*} \leq 0$ holds. (let $k^* = \infty$ if no such integer exists, although we show that such k^* exists).

Lemma 2. For each $k < k^*$, we have $\overline{z}_k < \overline{z}_{k+1}$, $\underline{z}_k > \underline{z}_{k+1}$, $\overline{z}_k \geq (1+\varepsilon)^k$, and $\underline{z}_k \leq (1-\varepsilon)^k$.

Proof. The monotonicity of $\overline{z}_k, \underline{z}_k$ is obvious from the definition, so we omit it.

The rest of the proof is by induction.[3] For $k = 1$, $\overline{z}_1 = 1+\varepsilon$ and $\underline{z}_1 = 1-\varepsilon$, and hence the claim is satisfied.

Fix $k < k^*$. Suppose that, up to $k-1$, we have the desired inequalities. Then for k,

$$\begin{aligned} \overline{z}_k &= \frac{1}{\underline{z}_{k-1}} + \varepsilon \\ &\geq \frac{1}{(1-\varepsilon)^{k-1}} + \varepsilon, \end{aligned}$$

[3]We thank Mamiko Yamashita for her suggestion of the proof idea.

and therefore,

$$\begin{aligned}
\overline{z}_k - (1+\varepsilon)^k &\geq \frac{1}{(1-\varepsilon)^{k-1}} + \varepsilon - (1+\varepsilon)^k \\
&= \frac{1}{(1-\varepsilon)^k}[1-\varepsilon+\varepsilon(1-\varepsilon)^k - (1-\varepsilon^2)^k] \\
&= \frac{1}{(1-\varepsilon)^k}[1-(1-\varepsilon^2)^k - \varepsilon(1-(1-\varepsilon)^k)],
\end{aligned}$$

where, because $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + ab^{k-2} + b^{k-1})$ for any real numbers a, b, we have

$$\begin{aligned}
1-(1-\varepsilon^2)^k &= \varepsilon^2[1+(1-\varepsilon^2)+(1-\varepsilon^2)^2+\ldots+(1-\varepsilon^2)^{k-1}], \\
1-(1-\varepsilon)^k &= \varepsilon[1+(1-\varepsilon)+(1-\varepsilon)^2+\ldots+(1-\varepsilon)^{k-1}],
\end{aligned}$$

and thus, $1-(1-\varepsilon^2)^k \geq \varepsilon(1-(1-\varepsilon)^k)$. Therefore, $\overline{z}_k - (1+\varepsilon)^k \geq 0$.

Similarly,

$$\begin{aligned}
\underline{z}_k &= \frac{1}{\overline{z}_{k-1}} - \varepsilon \\
&\leq \frac{1}{(1+\varepsilon)^{k-1}} - \varepsilon,
\end{aligned}$$

and therefore,

$$\begin{aligned}
\underline{z}_k - (1-\varepsilon)^k &\leq \frac{1}{(1+\varepsilon)^{k-1}} - \varepsilon - (1-\varepsilon)^k \\
&= \frac{1}{(1+\varepsilon)^k}[1+\varepsilon-\varepsilon(1+\varepsilon)^k - (1-\varepsilon^2)^k] \\
&= \frac{1}{(1+\varepsilon)^k}[1-(1-\varepsilon^2)^k - \varepsilon((1+\varepsilon)^k - 1)] \\
&= \frac{\varepsilon^2}{(1+\varepsilon)^k}[(1+(1-\varepsilon^2)+\ldots+(1-\varepsilon^2)^{k-1}) - (1+(1+\varepsilon)+\ldots+(1+\varepsilon)^{k-1})] \\
&\leq 0.
\end{aligned}$$

□

Because $\underline{z}_k = \frac{1}{\overline{z}_{k-1}} - \varepsilon$ and $\overline{z}_{k-1}$ is divergent, there is an integer k^* such that $\underline{z}_k \leq 0$ for all $k \geq k^*$.

In the following, we assume that, at k^*, we exactly have $\underline{z}_{k^*} = 0$. A similar argument holds true for the other case with $\underline{z}_{k^*} < 0$.[4]

If $\underline{z}_{k^*} = 0$, then it means that, for any $\theta > 0$ and $e \in (0, \theta]$, there exists a "level-k" type of the sender with $k \leq k^*$ such that his ability is θ and he plays e. Therefore, in state θ, any effort below θ is played by some type of the sender.

At level $k^* + 1$, we have $\overline{z}_{k^*+1} = \infty$. Therefore, for any $\theta > 0$ and any $e > \theta$, there exists a "level-k type of the sender with $k \leq k^* + 1$ such that his ability is θ and he plays e. Therefore, any effort above θ is also possible.

Accordingly, any wage level is also possible for any given θ. □

In conclusion, hypothesizing a fully-revealing and differentiable equilibrium when $d = 0$ is common knowledge leads to a similar no-prediction result as in the cheap-talk case.

References

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Spence, M. (1973): "Job Market Signaling," *Quarterly Journal of Economics*, 87(3), 355–374.

[4]In case $\underline{z}_{k^*} < 0$, there exists $z_{k^*} = 0 \in [\underline{z}_{k^*}, \bar{z}_{k^*}]$. All the arguments below would work by using z_{k^*} instead of $\underline{z}_{k^*}$.